MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

Approximating the Sum of Correlated Lognormal or Lognormal-Rice Random Variables

Neelesh Mehta, Andreas Molisch, Jingxian Wu, Jin Zhang

TR2006-010   June 2006

Abstract

A simple and novel method is presented to approximate by the lognormal distribution the probability density function of the sum of correlated lognormal random variables. The method is also shown to work well for approximating the distribution of the sum of lognormal-Rice or Suzuki random variables by the lognormal distribution. The method is based on matching a low-order Gauss-Hermite approximation of the moment-generating function of the sum of random variables with that of a lognormal distribution at a small number of points. Compared with methods available in the literature such as the Fenton-Wilkinson method, Schwartz-Yeh method, and their extensions, the proposed method provides the parametric flexibility to address the inevitable trade-off that needs to be made in approximating different regions of the probability distribution function.

ICC 2006

Approximating the Sum of Correlated Lognormal or Lognormal-Rice Random Variables

Neelesh B. Mehta[‡], *Member, IEEE*, Andreas F. Molisch[‡], *Fellow, IEEE*,
Jingxian Wu[†], and Jin Zhang[‡], *Senior Member, IEEE*

***Abstract*— A simple and novel method is presented to approximate by the lognormal distribution the probability density function of the sum of correlated lognormal random variables. The method is also shown to work well for approximating the distribution of the sum of lognormal-Rice or Suzuki random variables by the lognormal distribution. The method is based on matching a low-order Gauss-Hermite approximation of the moment-generating function of the sum of random variables with that of a lognormal distribution at a small number of points. Compared with methods available in the literature such as the Fenton-Wilkinson method, Schwartz-Yeh method, and their extensions, the proposed method provides the parametric flexibility to address the inevitable trade-off that needs to be made in approximating different regions of the probability distribution function.**

I. Introduction

The sum of lognormal random variables (RV) often occurs in wireless systems in signal to interference plus noise ratio (SINR), outage analysis, received signal power in a multipath environment, etc., [1, Chp. 3], [2]. Therefore, considerable efforts have been devoted to analyze the statistical properties of the lognormal sum. While exact closed-form expressions for its probability density function (pdf) are unknown, several analytical approximation methods have been proposed in the literature [3]–[8]. The lognormal sum pdf is approximated by a single lognormal pdf by Fenton-Wilkinson (F-W) [3], Schwartz-Yeh (S-Y) [4], and Beaulieu-Xie [5] methods. Bounds on the lognormal sum cumulative distribution function (CDF) have also been proposed in [1], [6]; but, these do not directly provide the form and parameters of the approximating pdf.

While independent lognormal RVs were the primary concern when the above methods were first proposed, extensions of the F-W [9], [10], S-Y [11], and Schleher's [9] methods have been proposed to handle the sum of correlated lognormal RVs. Correlated lognormal RVs, which are the focus of this paper, are of great interest because the shadowing of inter-cell interferers in a cellular system is correlated, with a typical site-to-site correlation coefficient of 0.5 [12].

The various methods described above all have their advantages and shortcomings, and none is unquestionably better than the others [9]. The S-Y method and its extension to the correlated case cannot accurately estimate small values of the complementary cumulative distribution function (CCDF) [13], while the F-W method and its extension cannot accurately estimate small values of the CDF. The outage probability bounds derived in [8] can also handle the correlated case, but they do not directly provide the form and parameters of the approximating pdf, and can be quite loose in some cases.

This paper presents a general method that uses the moment generating function (MGF) as a tool to approximate the sum of *correlated lognormal* RVs by a single lognormal RV. The method is an extension of an approach that was used recently for approximating the sum of *independent* lognormal RVs by a single lognormal RV [14]. The proven permanence of the lognormal pdf when the number of summands approaches infinity lends credence to such an approach [15], [16]. As we show, the method overcomes the shortcomings of the previous approaches by providing the parametric flexibility to make the inevitable trade-off in accurately matching different portions of the pdf.

We show that the proposed method can also approximate the sum of independent *Suzuki* RVs [17], [18, Chp. 5] and, more generally, the sum of *lognormal-Rice* RVs by a single lognormal RV. Such sums arise, for example, in instantaneous co-channel interference power calculations in which Rayleigh or Ricean fading along with lognormal shadow fading need to be accounted for. Approximating a sum of RVs by a lognormal RV has great utility in analysis because the ratio or product of two lognormal RVs remains a lognormal RV.

That the characteristic function, which is a special case of the MGF, can be used to find the approximating parameters has also been recognized by Beaulieu-Xie [5] and Barakat [15]. However, their methods require extremely accurate numerical computation at a sufficiently large number of points, and are quite involved. Moreover, they are fundamentally limited to the case in which the lognormal RVs are independent.

The paper is organized as follows: Section II motivates and defines the proposed method for approximating the sum of correlated lognormal RVs by a single lognormal RV. Section III extends the method to approximate the sum of Suzuki or, more generally, lognormal-Rice RVs by a single lognormal RV. Numerical examples are used in Section IV to compare it with other methods and to demonstrate its accuracy. The conclusions follow in Section V.

The authors are with the [‡]Mitsubishi Electric Research Labs (MERL), 201 Broadway, Cambridge, MA 02139, USA and [†]Dept. of Engineering Science, Sonoma State University, Rohnert Park, CA 94928, USA. (Email: mehta@merl.com, jingxian.wu@sonoma.edu, {molisch, jzhang}@merl.com). Andreas F. Molisch is also at the Dept. of Electroscience, Lund University, Sweden.

II. Sum of Correlated Lognormal RVs

A. Motivation of Proposed Method

Let $Y_1, \ldots, Y_K$ be K lognormal RVs with the joint distribution $p_{Y_1,\ldots,Y_K}(y_1, \ldots, y_K)$, with marginal pdfs denoted by $p_{Y_i}(x)$, for $1 \leq i \leq K$, respectively. Then each lognormal RV, Y_i, can be written as $10^{0.1X_i}$, where X_i is a Gaussian RV with mean μ_{X_i} dB and standard deviation σ_{X_i} dB.

The F-W method, the S-Y method, and the extensions proposed in the literature to apply these two methods to the correlated lognormal sum case are special instances of the following general system of equations:

$$\int_0^\infty f_m(y) p_Y(y) dy = \int_0^\infty f_m(y) p_{(\sum_{i=1}^K Y_i)}(y) dy, \quad (1)$$

where m equals 1 or 2, $f_1(.)$ and $f_2(.)$ are weight functions, and Y is the approximating lognormal RV [14].

The F-W method matches the mean and variance of the lognormal sum and the approximating lognormal RV, and thus uses the weight functions $f_1(y) = y$ and $f_2(y) = (y - \mu_Y)^2$, where μ_Y is the mean of Y. As both of these functions monotonically increase with y, approximation errors in the tail portion (large values of argument) of the lognormal sum pdf are penalized more, which explains why the F-W method tracks the tail portion well. On the other hand, the S-Y method, which matches the mean and variance of the logarithms of the lognormal sum and the approximating lognormal RV, employs the weight functions $f_1(y) = \log_{10} y$ and $f_2(y) = (10\log_{10} y - \mu_X)^2$, where μ_X is the mean of the Gaussian RV $X = 10\log_{10} Y$. Due to the singularity of $\log_{10} y$ at $y = 0$, mismatches near the origin are severely penalized by both these weight functions. Compared to the F-W method, the S-Y method accords a lower penalty to errors in the pdf tail. For these reasons, it does a better job tracking the head portion (small values of argument) of the pdf, but not its tail. Note that both the F-W and the S-Y methods use fixed weight functions and offer no way of overcoming their respective shortcomings.

The MGF, $\Psi_Y(s)$, of a random variable, Y, which is defined as

$$\Psi_Y(s) = \int_0^\infty \exp(-sy) p_Y(y) dy, \quad (\mathrm{Re}(s) \geq 0), \quad (2)$$

can also be interpreted as the weighted integral of the pdf, with the weight function as the exponential function $\exp(-sy)$, which monotonically decreases in magnitude as Re(s) increases. Here, Re(s) denotes the real part of the complex number s. The key difference compared to the previous methods, is that varying s provides a mechanism for adjusting, as required, the penalties allocated to errors in the head and tail portions of the lognormal sum pdf. For independent lognormal RVs, it possesses the additional advantage that the MGF of the sum is the product of the individual MGFs. We shall restrict our attention to real s in this paper. Varying s from 0 to ∞ still affords considerable flexibility, and is sufficient for our problem.

Based on the above motivation, this paper proposes a method that matches, at two points, the MGF of the sum of correlated lognormal RVs with the MGF of the single approximating lognormal RV. For this, we now derive expressions for the MGF of a lognormal RV and the sum of correlated lognormal RVs.

B. Lognormal MGF

No general closed-form expression for the MGF of the lognormal pdf is available. However, for real s, it can be readily expressed by a series expansion based on Gauss-Hermite integration. The MGF of a lognormal RV, Y, for real s, can be written as

$$\begin{aligned}\Psi_Y(s) &= \int_0^\infty \exp(-sy) \frac{\xi}{y\sigma_X\sqrt{2\pi}} \exp\left[-\frac{(\xi\log_e y - \mu_X)^2}{2\sigma_X^2}\right] dy, \\ &= \sum_{n=1}^N \frac{w_n}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right] + R_N,\end{aligned} \quad (3)$$

where μ_X and σ_X are the mean and standard deviation of the Gaussian RV $X = 10\log_{10} Y$. In (3), which is the Gauss-Hermite series expansion of the MGF, N is the Hermite integration order, R_N is a remainder term that decreases as N increases, and $\xi = 0.1\log_e 10$. The weights, w_n, and abscissas, a_n, for N up to 20 are tabulated in [19, Tbl. 25.10].

From (3), we define the Gauss-Hermite representation, $\widehat{\Psi}_Y(s; \mu_X, \sigma_X)$, of the lognormal MGF by removing R_N as follows:

$$\widehat{\Psi}_Y(s; \mu_X, \sigma_X) \triangleq \sum_{n=1}^N \frac{w_n}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right]. \quad (4)$$

C. MGF of Sum of Correlated Lognormal RVs

We now find the MGF of the sum of K correlated lognormal RVs, $\{Y_i\}_{i=1}^K$, with corresponding Gaussian RVs, $\{X_i\}_{i=1}^K$. The K Gaussian RVs, $X_i = 10\log_{10} Y_i$, $i = 1, \ldots, K$, follow the joint distribution

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{K/2}|\mathbf{C}|^{1/2}} \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})^\dagger \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right), \quad (5)$$

where $\mathbf{C}$ is the covariance matrix, $\boldsymbol{\mu}$ is the vector of means, $|.|$ denotes the determinant, and $(.)^\dagger$ denotes the Hermitian transpose. The MGF of $Y_1 + \cdots + Y_K$ can be written as:

$$\begin{aligned}&\Psi^{(c)}_{(\sum_{k=1}^K Y_k)}(s) \\ &= \int_{-\infty}^\infty \frac{1}{(2\pi)^{K/2}|\mathbf{C}|^{1/2}} \prod_{i=1}^K \exp\left(-s\left[\exp\left(\frac{x_i}{\xi}\right)\right]\right) \\ &\qquad \times \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})^\dagger \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right) d\mathbf{x}. \quad (6)\end{aligned}$$

Let $\mathbf{C}_{\text{sq}}$ be the square root of the covariance matrix $\mathbf{C}$, *i.e.*, $\mathbf{C} = \mathbf{C}_{\text{sq}}\mathbf{C}_{\text{sq}}^\dagger$. In general, if the eigen-decomposition of $\mathbf{C}$ is $\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\dagger$, where $\mathbf{U}$ is the eigen-space of $\mathbf{C}$ and the diagonal matrix $\boldsymbol{\Lambda}$ contains the eigen-values of $\mathbf{C}$, then $\mathbf{C}_{\text{sq}} = \mathbf{U}\boldsymbol{\Lambda}^{1/2}$.

Using the decorrelating transformation $\mathbf{x} = \sqrt{2}\mathbf{C}_{\text{sq}}\mathbf{z} + \boldsymbol{\mu}$, x_k is given by

$$x_k = \sqrt{2}\sum_{j=1}^{K} c'_{kj} z_j + \mu_k, \quad k = 1, \ldots, K, \tag{7}$$

where c'_{kj} is the $(k,j)^{\text{th}}$ element of $\mathbf{C}_{\text{sq}}$. Therefore, (6) becomes

$$\Psi^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{1}{\pi^{K/2}} \exp\left(-\mathbf{z}^{\dagger}\mathbf{z}\right) \times \prod_{k=1}^{K} \exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=1}^{K} c'_{kj} z_j + \frac{\mu_k}{\xi}\right)\right]\right) d\mathbf{z}. \tag{8}$$

As before, taking the Gauss-Hermite expansion with respect to z_1 yields

$$\Psi^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{1}{\pi^{(K-1)/2}} \exp\left(-\sum_{i=2}^{K}|z_i|^2\right)\left(\sum_{n_1=1}^{N}\frac{w_{n_1}}{\sqrt{\pi}} \times \prod_{k=1}^{K}\exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=2}^{K} c'_{kj} z_j + \frac{\sqrt{2}}{\xi}c'_{k1}a_{n_1} + \frac{\mu_k}{\xi}\right)\right]\right)\right) \times dz_2 \ldots dz_K + R_N^{(1)}, \tag{9}$$

where $R_N^{(1)}$ is the remainder term. Proceeding in a similar manner for $z_2, \ldots, z_K$, we get

$$\Psi^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s) = \sum_{n_K=1}^{N}\cdots\sum_{n_1=1}^{N}\frac{w_{n_1}\ldots w_{n_K}}{\pi^{K/2}} \times \prod_{k=1}^{K}\exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{l=1}^{K} c'_{kl}a_{n_l} + \frac{\mu_k}{\xi}\right)\right]\right) + R_N^{(K)}, \tag{10}$$

where $R_N^{(K)}$ is the final remainder term. Rearranging the terms and dropping $R_N^{(K)}$ results in the following Gauss-Hermite representation, $\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s;\boldsymbol{\mu},\mathbf{C})$, for the MGF of the correlated lognormal sum:

$$\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s;\boldsymbol{\mu},\mathbf{C}) \triangleq \sum_{n_1=1}^{N}\cdots\sum_{n_K=1}^{N}\left[\prod_{k=1}^{K}\frac{w_{n_k}}{\sqrt{\pi}}\right] \times \exp\left(-s\sum_{k=1}^{K}\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=1}^{K} c'_{kj}a_{n_j} + \frac{\mu_k}{\xi}\right)\right]\right). \tag{11}$$

The above functional form also mimics the following desirable property of the MGF: For independent lognormal RVs, it is given by the product of the Gauss-Hermite MGF approximations of the individual summands, *i.e.*, $\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s;\boldsymbol{\mu},\mathbf{C}) = \prod_{k=1}^{K}\widehat{\Psi}_{Y_k}(s;\mu_k,\sigma_k)$.

D. Proposed Method

The sum, $Y_1 + \cdots + Y_K$, of K correlated lognormal RVs is approximated by a single lognormal RV, Y, whose parameters, μ_X and σ_X, are found by matching their respective Gauss-Hermite representations of the MGF at $s = s_1$ and s_2. This leads to the following system of two equations:

$$\sum_{n=1}^{N}\frac{w_n}{\sqrt{\pi}}\exp\left[-s_i\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right] = \widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s_i;\boldsymbol{\mu},\mathbf{C}), \quad \text{at } i = 1 \text{ and } 2, \tag{12}$$

where $\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s;\boldsymbol{\mu},\mathbf{C})$ is given by (11).

Note that the right hand side of the above two equations is a constant that needs to be calculated only twice at s_1 and s_2. These non-linear equations in μ_X and σ_X can be readily solved numerically using standard functions such as `fsolve` in Matlab and `NSolve` in Mathematica.

While the accuracy of the MGF approximation increases as the Hermite integration order, N, increases, accurate estimates of μ_X and σ_X can be obtained even for small N. This is because the form of (12) makes them insensitive to errors in the MGF approximation. We have found $N = 12$ to be more than sufficient for many different system parameters, which is small compared to the 20 to 40 terms required to achieve numerical accuracy in each iterative step in the S-Y method [20].

Increasing s penalizes more the errors in approximating the head portion of the lognormal sum pdf, while reducing s penalizes errors in the tail portion, as well. The inevitable trade-off that needs to be made in approximating both the head and tail portions of the pdf can now be done depending on the application. For example, when the lognormal sum arises because various signal components add up and the main performance metric is a small signal outage probability, the head of the CDF needs to be computed accurately. On the other hand, when the lognormal sum appears in the denominator only, for example, when the co-channel interferer powers add up in the SINR calculation, it is the tail portion of the sum pdf or, equivalently, the tail portion of the CCDF that needs to be calculated accurately to analyze outage. The proposed method can handle both of these applications by using different matching pairs, (s_1, s_2). As a general rule, larger values of s are used to match the CDF while smaller values of s are used to match the CCDF. This is elaborated upon in Section IV.

For the special case of the sum of two zero-mean lognormal RVs with correlation coefficient ρ and variance σ dB, the expression for $\widehat{\Psi}^{(c)}_{(Y_1+Y_2)}(s)$ in (11) simplifies to the following closed-form:

$$\widehat{\Psi}^{(c)}_{(Y_1+Y_2)}(s) \triangleq \sum_{n=1}^{N}\sum_{m=1}^{N}\frac{w_n w_m}{\pi}\exp\left(-s\left[\exp\left(\frac{\sqrt{2}\sigma a_m}{\xi}\right) + \exp\left(\frac{\sqrt{2(1-\rho^2)}\sigma a_n + \sqrt{2}\sigma\rho a_m}{\xi}\right)\right]\right). \tag{13}$$

III. Sum of Independent Suzuki or Lognormal-Rice Random Variables

The lognormal-Rice RV is the product of a lognormal RV and a Ricean-fading RV, and can be written as

$$W = Z\ 10^{0.1X}, \tag{14}$$

where Z is a Ricean RV with unit power and the Rice-coefficient κ. As mentioned, it models the signal power due to shadowing and the presence of a line of sight component along with a large number of non-line of sight components. The Suzuki RV, which is the product of a lognormal RV and a Rayleigh fading RV, is a special case of the lognormal-Rice RV, and occurs when $\kappa = 0$.

Given the desirable properties of the lognormal RV, techniques have been proposed in the literature to approximate the sum of K independent lognormal-Rice RVs by a single lognormal RV. An extension of the F-W-based moment matching method was proposed in [21]. Another technique is a two-step approximation process in which each of the lognormal-Rice or Suzuki RVs is first approximated by a lognormal RV (by equating the means and variances), and then the sum of the lognormal RVs is again approximated by a single lognormal RV using the F-W or the S-Y methods. The sum of Suzuki RVs has instead been approximated by another Suzuki RV in [22]. Exact formulae are available in the literature that express the outage probability of a sum of lognormal-Rice RVs in the form of a single integral, which is evaluated numerically [23], [24]. However, these do not address the problem of approximating by a single lognormal RV.

We now extend the proposed method to approximate the sum of K independent lognormal-Rice RVs, $S_1+\cdots+S_K$, by a single lognormal RV, Y. For this, we first need an expression for the MGF of the lognormal-Rice RVs. Using Gauss-Hermite integration and neglecting the remainder term result in the following MGF approximation for the k^{th} RV, S_k:

$$\begin{aligned}
&\widehat{\Psi}_{S_k}(s;\mu_k,\sigma_k,\kappa_k) \\
&\quad \triangleq \sum_{i=1}^{K} \frac{w_i(1+\kappa_k)/\sqrt{\pi}}{1+\kappa_k+s\exp\left(\frac{\sqrt{2}\sigma_k a_i}{\xi}+\frac{\mu_k}{\xi}\right)} \\
&\quad \times \exp\left(-\frac{s\kappa_k \exp\left(\frac{\sqrt{2}\sigma_k a_i}{\xi}+\frac{\mu_k}{\xi}\right)}{1+\kappa_k+s\exp\left(\frac{\sqrt{2}\sigma_k a_i}{\xi}+\frac{\mu_k}{\xi}\right)}\right),
\end{aligned} \tag{15}$$

where μ_k and σ_k are the logarithmic mean and logarithmic standard deviation of the shadowing component, and κ_k is the Rice factor [25]. As before, we restrict our attention to real s, which still provides considerable freedom in adjusting the integral weights.

Therefore, the mean μ_X dB and variance σ_X dB – the defining parameters of Y – are determined by matching, as before, the Gauss-Hermite representations of the MGFs at two points. This leads to the following system of two equations:

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma_X a_n+\mu_X}{\xi}\right)\right] \\
&\qquad = \prod_{k=1}^{K} \widehat{\Psi}_{S_k}(s_i;\mu_k,\sigma_k,\kappa_k), \quad \text{at } i=1 \text{ and } 2.
\end{aligned} \tag{16}$$

The right hand side term, $\prod_{k=1}^{K} \widehat{\Psi}_{S_k}(s_i;\mu_k,\sigma_k,\kappa_k)$, is a number, which consists entirely of known quantities, and needs to be evaluated only twice at s_1 and s_2 using (15).

It can be easily seen that the mixture case, in which not all of the RVs follow the same type of distribution, can also be readily approximated by a single lognormal RV by means of the proposed method by using the corresponding Gauss-Hermite representations for the MGFs of lognormal or lognormal-Rice or Suzuki RVs.

IV. Numerical Examples

In the examples below, we plot the CDFs and CCDFs of the sum pdf obtained from Monte Carlo simulations and compare them with those obtained using the various lognormal approximation methods. Small values of the CDF reveal the accuracy in tracking the head portion of the pdf, while small values of the CCDF reveal the accuracy in tracking the tail portion of the pdf. We also show that the same values of s_1 and s_2 work well for a variety of system parameters.

A. Sum of Correlated Lognormal RVs

We first consider the sum of K correlated lognormal RVs. As an example, the covariance matrix takes the functional form:

$$\mathbf{C} = \begin{bmatrix} 1 & \rho & \cdots & \rho^{K-1} \\ \rho & 1 & \cdots & \rho^{K-2} \\ & & \ddots & \\ \rho^{K-1} & \rho^{K-2} & \cdots & 1 \end{bmatrix}, \tag{17}$$

where ρ is the correlation coefficient between any two successive RVs. The logarithmic mean of each of the RVs is 0 dB.

The CDF of the sum of 4 correlated lognormal RVs, obtained through simulations, is plotted in Fig. 1 for the case in which all the constituent RVs have $\sigma = 8$ dB. The CDF of the lognormal RV with parameters estimated using the proposed method (using (12)) is compared with the CDFs obtained from the F-W and S-Y extensions. The comparison is made for two different values of the correlation coefficient: $\rho = 0.3$ and $\rho = 0.7$. For the proposed method, the MGFs are matched at $s_1 = 0.2$ and $s_2 = 1.0$. It can be seen that the proposed method can accurately track the CDF of the correlated lognormal sum, and is marginally better than the S-Y extension method. The F-W extension is the least accurate of all the methods.

Figure 2 plots the corresponding CCDF curves. As the tail of the pdf needs to be matched accurately to match the CCDF curves, smaller values of s are used in the proposed method: $s_1 = 0.001$ and $s_2 = 0.005$. It can be seen that the accuracy of the proposed method is comparable to that of the F-W extension, and is significantly better than the S-Y extension, which is the least accurate of all the methods. As expected, when the constituent lognormal RVs are highly correlated, all the methods can accurately track the CDF and the CCDF.

Figure 3 plots the CDF of the sum of different numbers of correlated lognormal RVs with $\rho = 0.3$ and shows that the proposed method is accurate in all cases. As in Fig. 1, the MGF is matched at $s_1 = 0.2$ and $s_2 = 1.0$. As expected, as K decreases, the accuracy of all methods improves.

B. Sum of Suzuki or Lognormal-Rice RVs

The effect of the Rice-coefficient, κ, is examined in Figure 4, which plots the CDF of the sum of 6 lognormal-Rice RVs with a lognormal variance of $\sigma = 6$ dB and a logarithmic mean of $\mu = 0$ dB. Also plotted is the CDF of the lognormal distribution obtained from the proposed method (using (16)). Figure 5 plots the corresponding CCDF. As was done in Section IV-A, the MGFs are again matched at $s_1 = 0.2$ and $s_2 = 1.0$ to track the CDF and at $s_1 = 0.001$ and $s_2 = 0.005$ to track the CCDF. We can see that the CDF and the CCDF can both be accurately approximated. As κ decreases, the accuracy of the approximation by a lognormal improves.

Figure 6 evaluates the impact of varying the number of summands, K, on the accuracy of the lognormal approximation It plots the CDF of the lognormal approximations obtained using the proposed method and the F-W-based method and compares them with Monte Carlo simulation results. It can be seen that the proposed method accurately approximates the sum of K Suzuki RVs by a single lognormal RV for $K = 2, 4$, and 8 RVs. As before, the MGFs are matched at $s_1 = 0.2$ and $s_2 = 1.0$.

V. Conclusions

We proposed a simple and novel method to approximate the sum of several correlated lognormal random variables with a single lognormal random variable. The method was also shown to accurately model the sum of independent lognormal-Rice (or Suzuki) RVs by a single lognormal RV. The results led to the important and useful observation that the points at which the Gauss-Hermite representations of the MGF are matched to obtain an accurate lognormal approximation remain the same over a wide range of system parameters. Specifically, it was shown that matching at $s_1 = 0.2$ and $s_2 = 1.0$ accurately modeled the CDF of the sum of correlated lognormal or lognormal-Rice RVs over a wide range of lognormal variances, correlations, Rice-coefficients, and for different numbers of summands. Similarly, matching at $s_1 = 0.001$ and $s_2 = 0.005$ accurately tracked the CCDF of the sum of RVs.

By using an approximate and short Gauss-Hermite expansion of the lognormal MGF, the proposed method circumvents the requirement for very precise numerical computations at a large number of points. It is numerically stable and, as we show, accurate. The method was motivated by the interpretation of the MGF as a weighted integral of the pdf. It is a tool that provides the parametric flexibility needed to approximate, as accurately as required, different portions of the pdf.

References

[1] G. L. Stüber, *Principles of Mobile Communications*. Kluwer Academic Publishers, 1996.

[2] H. Liu, "Error performance of a pulse amplitude and position modulated ultra-wideband system over lognormal fading channels," *IEEE Commun. Lett.*, vol. 7, pp. 531–533, 2003.

[3] L. F. Fenton, "The sum of lognormal probability distributions in scatter transmission systems," *IRE Trans. Commun. Syst.*, vol. CS-8, pp. 57–67, 1960.

[4] S. Schwartz and Y. Yeh, "On the distribution function and moments of power sums with lognormal components," *Bell Syst. Tech. J.*, vol. 61, pp. 1441–1462, 1982.

[5] N. C. Beaulieu and Q. Xie, "An optimal lognormal approximation to lognormal sum distributions," *IEEE Trans. Veh. Technol.*, vol. 53, pp. 479–489, 2004.

[6] S. B. Slimane, "Bounds on the distribution of a sum of independent lognormal random variables," *IEEE Trans. Commun.*, vol. 49, pp. 975–978, 2001.

[7] D. C. Schleher, "Generalized Gram-Charlier series with application to the sum of log-normal variates," *IEEE Trans. Inform. Theory*, pp. 275–280, 1977.

[8] F. Berggren and S. Slimane, "A simple bound on the outage probability with lognormally distributed interferers," *IEEE Commun. Lett.*, vol. 8, pp. 271–273, 2004.

[9] A. A. Abu-Dayya and N. C. Beaulieu, "Outage probabilities in the presence of correlated lognormal interferers," *IEEE Trans. Veh. Technol.*, vol. 43, pp. 164–173, 1994.

[10] F. Graziosi, L. Fuciarelli, and F. Santucci, "Second order statistics of the SIR for cellular mobile networks in the presence of correlated co-channel interferers," in *Proc. VTC*, pp. 2499–2503, 2001.

[11] A. Safak, "Statistical analysis of the power sum of multiple correlated log-normal components," *IEEE Trans. Veh. Technol.*, vol. 42, pp. 58–61, 1993.

[12] "Spatial channel model for multiple input multiple output (MIMO) simulations," Tech. Rep. 25.996, 3rd Generation Partnership Project.

[13] A. A. Abu-Dayya and N. C. Beaulieu, "Comparison of methods of computing correlated lognormal sum distributions and outages for digital wireless applications," in *Proc. VTC*, pp. 175–179, 1994.

[14] J. Wu, N. B. Mehta, and J. Zhang, "A flexible lognormal sum approximation method," in *Proc. Globecom*, Nov. 2005.

[15] R. Barakat, "Sums of independent lognormally distributed random variables," *J. Opt. Soc. Am.*, vol. 66, pp. 211–216, 1976.

[16] W. A. Janos, "Tail of the distributions of sums of lognormal variates," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 299–302, 1970.

[17] H. Suzuki, "A statistical model for urban radio propagation," *IEEE Trans. Commun.*, vol. 25, pp. 673–677, 1977.

[18] A. F. Molisch, *Wireless Communications*. Wiley-IEEE Press, 2005.

[19] M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover, 9 ed., 1972.

[20] C.-L. Ho, "Calculating the mean and variance of power sums with two log-normal compoenents," *IEEE Trans. Veh. Technol.*, vol. 44, pp. 756–762, 1995.

[21] F. Graziosi and F. Santucci, "On SIR fade statistics in Rayleigh-lognormal channels," in *Proc. ICC*, pp. 1352–1357, 2002.

[22] J. E. Tighe and T. T. Ha, "On the sum of multiplicative chi-square-lognormal random variables," in *Proc. Globecom*, pp. 3719–3722, 2001.

[23] J.-P. M. Linnartz, "Exact analysis of the outage probability in multiple-user radio," *IEEE J. Select. Areas Commun.*, vol. 10, pp. 20–23, 1992.

[24] M. D. Austin and G. L. Stuber, "Exact co-channel interference analysis for log-normal shadowed Rician fading channels," *Electron. Lett.*, vol. 30, pp. 748–749, 1994.

[25] C. Tellambura and A. Annamalai, "An unified numerical approach for computing the outage probability for mobile radio systems," *IEEE Commun. Lett.*, vol. 3, pp. 97–99, 1999.

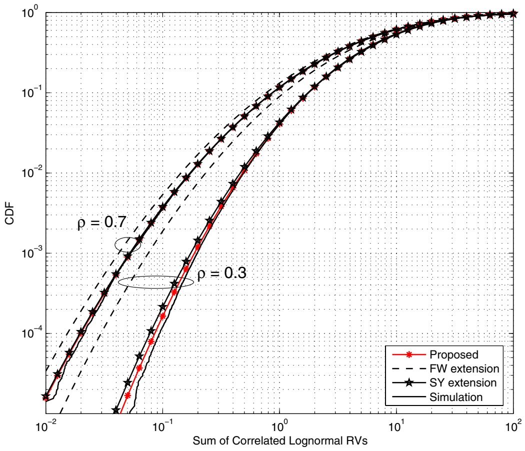

Fig. 1. Comparing the accuracy of the lognormal approximation techniques in tracking the CDF of the sum of correlated lognormal RVs for different ρ ($K = 4$, $\sigma = 8$) with $s_1 = 0.2$ and $s_2 = 1.0$ for the proposed method.

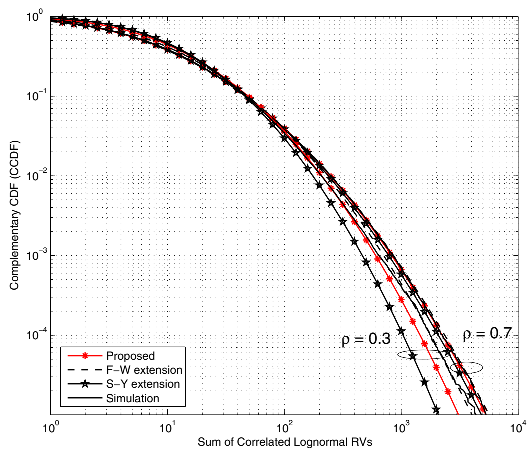

Fig. 2. Comparing the accuracy of the lognormal approximation techniques in tracking the CCDF for the sum of correlated lognormal RVs for different ρ ($K = 4$, $\sigma = 8$) with $s_1 = 0.001$ and $s_2 = 0.005$ for the proposed method.

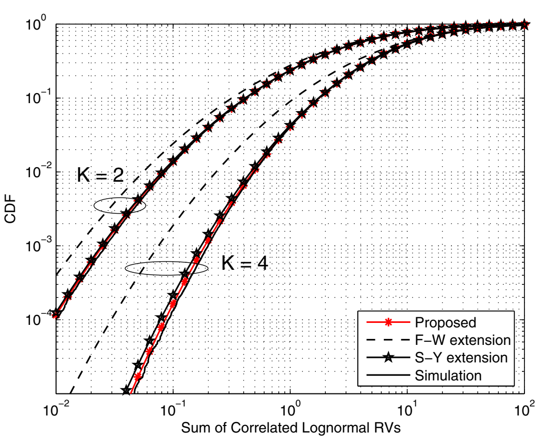

Fig. 3. Effect of number of summands, K, on the accuracy of approximating the CDF ($\sigma = 8$ dB and $\rho = 0.3$) with $s_1 = 0.2$ and $s_2 = 1.0$ for the proposed method.

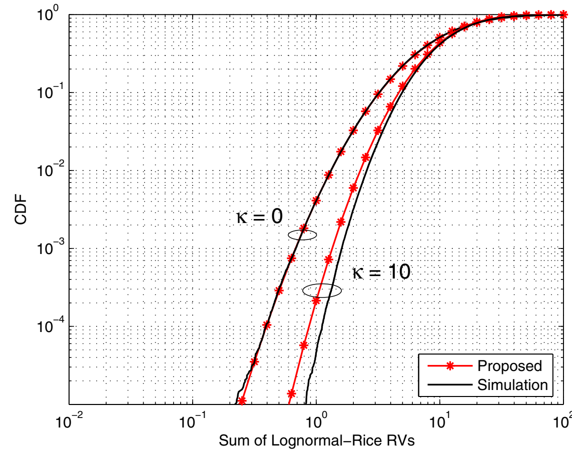

Fig. 4. Effect of Rice-factor, κ, on approximating the CDF of a sum of lognormal-Rice RVs by a lognormal RV using the proposed method with $s_1 = 0.2$ and $s_2 = 1.0$ ($\sigma = 6$ dB).

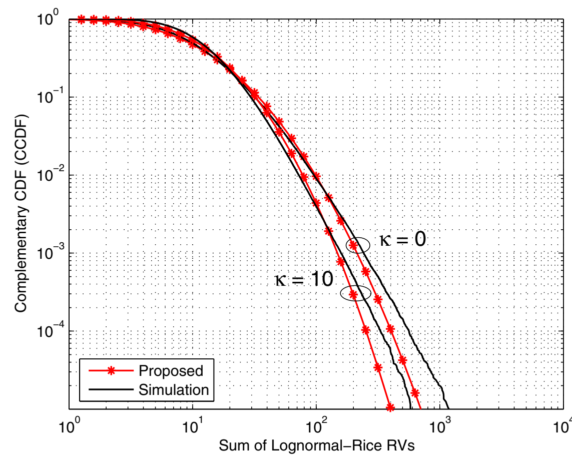

Fig. 5. Effect of Rice-factor, κ, on approximating the CCDF of a sum of lognormal-Rice RVs by a lognormal RV using the proposed method with $s_1 = 0.001$ and $s_2 = 0.005$ ($\sigma = 6$ dB).

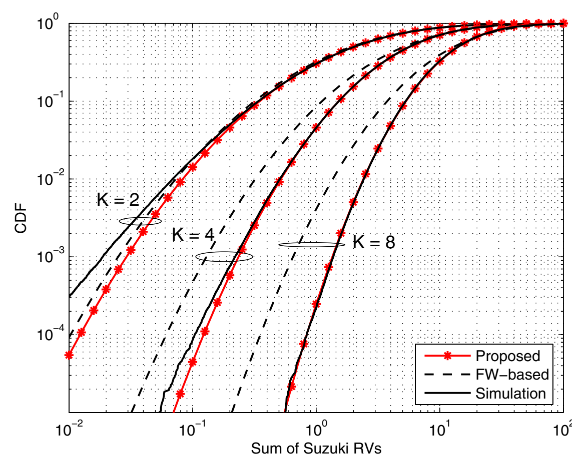

Fig. 6. Effect of number of Suzuki RVs on the accuracy of approximating the CDF for $\sigma = 6$ dB with $s_1 = 0.2$ and $s_2 = 1.0$ for the proposed method.