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***Index Terms*— LDPC codes, belief propagation, min-sum decoding.**

I. INTRODUCTION

Low-density parity-check (LDPC) codes [1] with BP decoding [2] achieve a remarkable error performance near to the Shannon limit [3]. Nevertheless it can become too complex for hardware implementation. By approximating the calculation at the check nodes with a simple minimum operation, the MS algorithm reduces the complexity of BP [4]. While MS is hardware efficient, its performance is often much worse than that of BP. It has been observed that the degradation due to MS decoding can be compensated by linear post processing (normalization or offset) of the messages delivered by check nodes [5]. Simulation results and density evolution analysis show that for decoding regular LDPC codes, normalized or offset MS with a single correction factor is sufficient to achieve performance close to that of BP decoding. For many irregular LDPC codes [6], however, conventional correction MS exhibits a large performance degradation compared to that of BP. In this paper, we present a 2-D normalized MS decoding of irregular LDPC codes. In 2-D normalized MS scheme, belief messages outgoing from both check and bit node processors are normalized. The normalization factor of a check (bit) node processor depends on the degree of that node. To circumvent brute force search which is intractable for most irregular LDPC codes of practical interests, parallel differential optimization and density evolution are used to obtain the 2-D optimal normalization factor pair. We also extend this idea to 2-D offset MS decoding.

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II. STANDARD BP

Suppose a binary (N, K) LDPC code is used with BPSK signaling over an AWGN channel. Let $\mathbf{H} = [H_{mn}]$ be the parity check matrix which defines this LDPC code. We denote $\mathcal{N}(m) = \{n : H_{mn} = 1\}$ and $\mathcal{M}(n) = \{m : H_{mn} = 1\}$. Let $U_{ch,n}$ be the log-likelihood ratio (LLR) of bit n. At iteration i, let $U_{mn}^{(i)}$ and $V_{mn}^{(i)}$ be the LLR of bit n which are sent from check node m to bit node n and from the bit node n to check node m, respectively. The check node and bit node processing steps in the standard LLR BP algorithm are carried out as follows [2]:

(i) Horizontal Step, for $1 \le n \le N$ and each $m \in \mathcal{M}(n)$:

$$U_{mn}^{(i)} = 2\tanh^{-1} \prod_{n' \in \mathcal{N}(m)\backslash n} \tanh \frac{V_{mn'}^{(i-1)}}{2}. \quad (1)$$

(ii) Vertical Step, for $1 \le n \le N$ and each $m \in \mathcal{M}(n)$:

$$V_{mn}^{(i)} = U_{ch,n} + \sum_{m' \in \mathcal{M}(n)\backslash m} U_{m'n}^{(i)}. \quad (2)$$

III. MS AND NORMALIZED MS DECODING

Taking advantage of the odd property of the function tanh()[1], MS simplifies the updating rule in check nodes by modifying (1) into

$$U_{mn}^{(i)} = \prod_{n' \in \mathcal{N}(m)\backslash n} \text{sgn}\left(V_{mn'}^{(i-1)}\right) \cdot \min_{n' \in \mathcal{N}(m)\backslash n} |V_{mn'}^{(i-1)}| \quad (3)$$

The normalized MS modifies the check node processing (3) as

$$U_{mn}^{i} \leftarrow \alpha \cdot U_{mn}^{i} \quad (4)$$

where α is a normalization factor, $0 < \alpha \le 1$.

IV. 2-D NORMALIZED MS

Another way to improve the performance of MS is to reprocess the LLR from bit nodes by modifying (2) into

$$V_{mn}^{(i)} = U_{ch,n} + \beta \cdot \sum_{m' \in \mathcal{M}(n)\backslash m} U_{m'n}^{(i)} \quad (5)$$

where β is a normalization factor. For regular LDPC codes, normalization factor is the same for each check (bit) node.

[1]This property implies $\text{sgn}\left(2\tanh^{-1}\prod_{n'}\tanh\frac{V_{mn'}}{2}\right) = \prod_{n'}\text{sgn}(V_{mn'})$.

In this case, bit node normalization (5) is equivalent to check node normalization. For irregular LDPC codes, the conventional normalized MS only reprocesses the LLRs outgoing from check nodes while keeping the bit node processor unchanged. The outgoing LLRs are normalized equally without considering the degree differences among the adjacent bit nodes in the graph representation of the code. However, intuitively, for bit nodes with different degrees, this same incoming LLR should play a different role in the outgoing LLRs. This suggests to apply normalization in both check and bit nodes. As conventionally, only outgoing LLR from check nodes (horizontal step) are normalized, we refer to it as one-dimensional normalized MS while we refer to this proposed approach as 2-D (both horizontal and vertical step) normalized MS decoding.

A. 2-D normalized MS

Consider an irregular LDPC code with degree distribution $\lambda(x) = \sum_{j=1}^{dv_{max}} \lambda_j x^{j-1}$ and $\rho(x) = \sum_{j=1}^{dc_{max}} \rho_j x^{j-1}$, which specify the degree distribution of bit nodes and check nodes, respectively. Let α_j be the normalization factor for check nodes with degree j, for $j = 1, 2, \ldots, dc_{max}$. Let β_j be the normalization factor for bit nodes with degree j, for $j = 1, 2, \ldots, dv_{max}$. Let $dv(n)$ be the degree of bit node n, for $n = 1, 2, \ldots, N$. Let $dc(m)$ be the degree of check node m, for $m = 1, 2, \ldots, M$. 2-D normalized MS decoding is specified by

(i) Horizontal Step, for $1 \le n \le N$ and each $m \in \mathcal{M}(n)$:

$$U_{mn}^{(i)} = \alpha_{dc(m)} \prod_{n' \in \mathcal{N}(m) \backslash n} \text{sgn}\left(V_{mn'}^{(i-1)}\right) \min_{n' \in \mathcal{N}(m) \backslash n} |V_{mn'}^{(i-1)}|$$

(ii) Vertical Step, for $1 \le n \le N$ and each $m \in \mathcal{M}(n)$:

$$V_{mn}^{(i)} = U_{ch,n} + \beta_{dv(n)} \cdot \sum_{m' \in \mathcal{M}(n) \backslash m} U_{m'n}^{(i)} \quad (6)$$

Note that if the code is either bit-regular or check-regular, only one dimensional normalization is needed (normalization factors still can be different for nodes with different weights).

B. Density evolution for 2-D normalized MS

It is straightforward to verify that 2-D normalized MS decoding satisfies the channel, check node, and bit node processing symmetric conditions and therefore its density evolution can be analyzed based on the all-zero transmitted codeword [3]. However there is no guarantee that the probability of error is non-increasing for 2-D normalized MS decoding [7]. In MS decoding, let $f_U^{(i)}(u)$ and $f_V^{(i)}(v)$ be the pdfs of LLRs from check and bit nodes, respectively. The density evolution of 2-D normalized MS is specified by

$$f_U^{(i)}(u) \leftarrow \sum_{j=1}^{dc_{max}} \frac{\rho_j}{\alpha_j} f_U^{(i)}\left(\frac{u}{\alpha_j}\right)$$

$$f_V^{(i)}(v) \leftarrow \sum_{j=1}^{dv_{max}} \frac{\lambda_j}{\beta_j} \mathcal{F}^{-}\left(\mathcal{F}(f_{U_{ch}}) \cdot \left(\mathcal{F}(f_U^{(i)})\right)^{j-1}\right)\left(\frac{v}{\beta_j}\right)$$

C. Parallel differential evolution for 2-D normalized MS

Let $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_{dc_{max}}\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \ldots, \beta_{dv_{max}}\}$ be the normalization factors (vectors) of check and bit nodes in 2-D normalization MS decoding, respectively. Given an irregular LDPC code with degree distribution pair (λ, ρ) and a normalization factor pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$, we can evaluate the threshold of this code with 2-D normalized MS decoding, by using density evolution. We search for the normalization factor pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$ which yields the largest noise threshold. This is a nonlinear minimization problem with continuous space parameters. The brute force search method becomes intractable when $dv_{max} \cdot dc_{max}$ is large. However an algorithm called parallel differential optimization has been shown to be efficient and robust. In [6], this technique has been successfully applied to construct irregular LDPC codes. In this paper, parallel differential optimization is used to generate the normalization factor pair of 2-D normalized MS decoding.

Given an irregular (dv_{max}, dc_{max}) LDPC code, the number of degrees of freedom for a normalization factor pair is $dv_{max} \cdot dc_{max}$. In differential evolution optimization, the population of each generation is often selected to be a multiple of the number of degrees of freedom. Let $L = T \cdot dv_{max} \cdot dc_{max}$ be the population in each generation, where T is a constant. Apparently, the larger the value of T, the higher the probability that a better result can be found, and correspondingly the higher the needed complexity. Let $(\boldsymbol{\alpha}_l^{(g)}, \boldsymbol{\beta}_l^{(g)})$ denote the l-th member of generation g. The procedure to find the optimal factor pair of 2-D normalized MS based on density evolution and parallel differential optimization is carried out as follows

1 Initialization: Set the maximum number of generations to G_{max} and the generation index $g = 0$. Randomly generate a set of pairs $\{(\boldsymbol{\alpha}_l^{(0)}, \boldsymbol{\beta}_l^{(0)})\}$ with cardinality L. For $l = 1, 2, \ldots, L$, run density evolution of 2-D normalized MS based on the factor pair $(\boldsymbol{\alpha}_l^{(0)}, \boldsymbol{\beta}_l^{(0)})$ and obtain the corresponding threshold $(\frac{E_b}{N_o})_l^{(0)}$. Then find the factor pair which yields the smallest $\frac{E_b}{N_o}$. We denote this best pair as $(\boldsymbol{\alpha}_{l_{best}}^{(0)}, \boldsymbol{\beta}_{l_{best}}^{(0)})$, where $l_{best} = \arg\min_{l=1}^{L} \left(\frac{E_b}{N_o}\right)_l^{(0)}$.

2 Mutation and test: Set $g \leftarrow g+1$. For $l = 1, 2, \ldots, L$, generate J distinct random numbers $\{r_j | 1 \le r_j \le L, r_j \ne l\}$, and generate the test vector

$$(\boldsymbol{\alpha}_l^{(g+1)}, \boldsymbol{\beta}_l^{(g+1)})_t = (\boldsymbol{\alpha}_{l_{best}}^{(g)}, \boldsymbol{\beta}_{l_{best}}^{(g)}) + \gamma \cdot \left(\sum_{\substack{1 \le j \le J \\ j:odd}} (\boldsymbol{\alpha}_{r_j}^{(g)}, \boldsymbol{\beta}_{r_j}^{(g)}) - \sum_{\substack{1 \le j \le J \\ j:even}} (\boldsymbol{\alpha}_{r_j}^{(g)}, \boldsymbol{\beta}_{r_j}^{(g)})\right)$$

where γ is a pre selected constant which controls the amplification of the differential variation. The use of differences between candidate vectors increases the variation, helping the iteration escaping from a local minimum. Then for $l = 1, 2, \ldots, L$, run density evolution of 2-D normalized MS based on the newly generated pair $(\boldsymbol{\alpha}_l^{(g+1)}, \boldsymbol{\beta}_l^{(g+1)})_t$ and obtain the corresponding threshold $(\frac{E_b}{N_o})_{l,t}^{(g+1)}$.

3 Compare and update: For $l = 1, 2, \ldots, L$, set

$$(\boldsymbol{\alpha}_l^{(g+1)}, \boldsymbol{\beta}_l^{(g+1)}) = \begin{cases} (\boldsymbol{\alpha}_l^{(g+1)}, \boldsymbol{\beta}_l^{(g+1)})_t & \text{if } (\frac{E_b}{N_o})_{l,t}^{(g+1)} \leq (\frac{E_b}{N_o})_l^{(g)} \\ (\boldsymbol{\alpha}_l^{(g)}, \boldsymbol{\beta}_l^{(g)}) & \text{otherwise} \end{cases}$$

and

$$\left(\frac{E_b}{N_o}\right)_l^{(g+1)} = \begin{cases} \left(\frac{E_b}{N_o}\right)_{l,t}^{(g+1)} & \text{if } \left(\frac{E_b}{N_o}\right)_{l,t}^{(g+1)} \leq \left(\frac{E_b}{N_o}\right)_l^{(g)} \\ \left(\frac{E_b}{N_o}\right)_l^{(g)} & \text{otherwise} \end{cases}$$

Then update the best factor pair of generation $g+1$ to $(\boldsymbol{\alpha}_{l_{best}}^{(g+1)}, \boldsymbol{\beta}_{l_{best}}^{(g+1)})$, where $l_{best} = \arg\min_{l=1}^{L} \left(\frac{E_b}{N_o}\right)_l^{(g+1)}$.

4 Stopping test and output: If G_{max} is reached or the smallest threshold of generations stops decreasing, output $(\boldsymbol{\alpha}_{l_{best}}^{(g+1)}, \boldsymbol{\beta}_{l_{best}}^{(g+1)})$. Otherwise, go to Step 2.

It is worth mentioning that the algorithm can be extended in a straightforward way if different normalization factors are used at different iterations, in which case the error performance of 2-D normalized MS decoding can be improved further.

V. 2-D Offset MS

The conventional (1-D) offset MS algorithm improves the accuracy of MS decoding by reducing the reliability of U_{mn} by a positive constant σ [5]

$$U_{mn} \leftarrow \text{sgn}(U_{mn}) \cdot \max(|U_{mn}| - \sigma, 0). \tag{7}$$

In 2-D offset MS decoding, belief messages coming from both check nodes and bit nodes are reduced by constants which are associated to the degrees of the corresponding check and bit nodes. More precisely, let σ_j's be the offset factors for check nodes with degree j, for $j = 1, 2, \ldots, d_{cmax}$, and let υ_j's be the offset factors for bit nodes with degree j, for $j = 1, 2, \ldots, d_{vmax}$. Then 2-D offset MS decoding is specified by (i) Horizontal Step, for $1 \leq n \leq N$ and each $m \in \mathcal{M}(n)$:

$$U_{mn}^{(i)} = \prod_{n' \in \mathcal{N}(m)\backslash n} \text{sgn}\left(V_{mn'}^{(i-1)}\right) \cdot \max\left(\min_{n' \in \mathcal{N}(m)\backslash n} \left|V_{mn'}^{(i-1)}\right| - \sigma_{dc(m)}, 0\right).$$

(ii) Vertical Step, for $1 \leq n \leq N$ and each $m \in \mathcal{M}(n)$:

$$V_{mn}^{(i)} = U_{ch,n} + \text{sgn}\left(S_{U_{m'n}^{(i)}}\right) \cdot \max\left(\left|S_{U_{m'n}^{(i)}}\right| - \upsilon_{dv(n)}, 0\right),$$

where

$$S_{U_{m'n}^{(i)}} = \sum_{m' \in \mathcal{M}(n)\backslash m} U_{m'n}^{(i)}.$$

The procedures to obtain the optimal offset factors for 2-D offset MS are similar to that in Section IV-C [8].

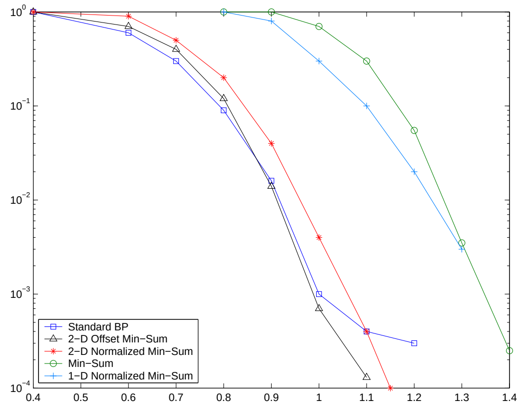

Fig. 1. WER of the standard BP, 2-D normalized Min-Sum, 2-D offset Min-Sum conventional normalized Min-Sum and Min-Sum algorithms for decoding a (16200,7200) irregular LDPC code (I_{max}=200).

VI. Simulation Results

Figure 1 depicts the word error rate (WER) of standard BP, 2-D normalized MS, 2-D offset MS, 1-D normalized MS and MS decodings of a (16200,7200) irregular LDPC code. The check and bit node distributions of this code are $\rho(x) = 0.00006x^2 + 0.14917x^3 + 0.29851x^4 + 0.44777x^5 + 0.10449x^6$ and $\lambda(x) = 0.00002 + 0.38803x + 0.31344x^2 + 0.29851x^7$, respectively. The thresholds computed by density evolution for standard BP, MS, 1-D normalized MS, 2-D normalized MS, and 2-D offset MS decodings are 0.77dB, 0.93dB, 0.90dB, 0.85dB, and 0.85dB, respectively. We observe that 2-D normalized and offset MS provide a comparable performance as BP and interestingly have a lower error floor than that of BP. We also observe that 2-D normalized and offset MS outperform 1-D normalized MS and MS by about 0.3dB. The normalization factor for 1-D normalized MS is $\alpha = 0.75$. The optimum normalization vectors of 2-D normalized MS decoding are $\boldsymbol{\alpha} = (1.00, 0.94, 0.92, 0.88, 0.86)$ and $\boldsymbol{\beta} = (1.00, 1.00, 0.91, 0.83)$. The optimum offset vectors for check and bit nodes in 2-D offset MS decoding are $\boldsymbol{\sigma} = (0.00, 0.26, 0.31, 0.35, 0.37)$ and $\boldsymbol{\upsilon} = (0.00, 0.00, 0.20, 0.31)$, respectively.

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