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Multiple Time Scaling of a Multi-Output Observer Form

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TR2010-017 May 2010

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IEEE Transactions on Automatic Control

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***Abstract*— Time scalings in the multi-output observer form for uncontrolled nonlinear continuous-time systems are considered in this paper. It is the multi-output version of an existing single-input result. Time scaling broadens the class of systems which admits an exact error linearization observer design by including time scaling transformations. The existence conditions of the time scaling transformation and the change of state coordinates to time-scaled observer form are provided.**

I. Introduction

We consider observer design for uncontrolled multi-output systems in state space form

$$\dot{x} = f(x), \quad y = h(x) \tag{1}$$

where $\dot{x}$ denotes dx/dt, $x \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a C^∞ output function. The well-established exact error linearization nonlinear observer design method uses an Observer Form (OF) to obtain stable LTI state estimate error dynamics in OF coordinates [1], [2]. Significant effort has been placed on extending this original work for single-output continuous-time systems, e.g. [3], [4], [5], [6], [7]. Recent work [8], [9] considers a generalization of exact error linearization which incorporates output dependent time scaling transformations for single-output nonlinear systems. Time scaling transformations lead to an additional degree of freedom when transforming the system to OF. Given the wide array of nonlinear observer design methods that have been developed, it is important to establish the useful properties of any approach. OF-based methods benefit from a relatively straightforward design procedure which exploits the target normal form and potentially larger regions of attraction with relatively low observer gains. Although other approaches such as [6], [10] consider different system classes, these properties of OF-based designs can make them attractive alternatives.

This paper considers a multi-output version of work in [8], [9]. In Section II we introduce the time-scaled multi-output observer form (TOF), and state the problem to be solved. In Section III we discuss the single and multiple time scaling transformation cases, propose the existence conditions of the TOF, compare the time scalings to output transformations, and investigate the design, implementation, and robustness of a TOF-based observers. Two numerical examples are given in Section IV to illustrate the construction of TOF coordinates, time scaling transformations, and the implementation of the proposed observer.

Manuscript received April 16, 2008; revised January 30, 2009 and July 24, 2009. Recommended by Associate Editor Alessandro Astolfi. This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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II. Problem Statement

Early work on time scaling for control design [11], [12], [13] enlarged the class of state feedback linearizable systems. To generalize the class of single-output systems which admits an OF, an output dependent time scaling transformation was introduced in [8]: $\dot{\tau} = s(y(t)) > 0,\ \tau(t_0) = \tau_0$, where $s(y(t))$ is a non-vanishing positive smooth function, called a time scale function (TSF). For multi-output systems, existence conditions for the multi-output observer form (OF) have been established in [4], [3]. This OF has the form

$$\dot{z} = Az + \gamma(y), \quad y = Cz,$$

where $A = \text{Blockdiag}\{A_1, \cdots, A_p\}$, $C = \text{Blockdiag}\{C_1, \cdots, C_p\}$, and $A_i = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{\lambda_i - 1} \\ 0 & \mathbf{0} \end{pmatrix}$, $C_i = (1, \mathbf{0})^T$ with $\mathbf{I}_k$ a $k \times k$ identity matrix.

Given time scaling transformations for each subsystem

$$\dot{\tau}_i = s_i(y(t)) > 0, \quad \tau_i(t_0) = \tau_{i0}, \tag{2}$$

we define the TOF as an OF in the time τ

$$\frac{\mathrm{d}z_i}{\mathrm{d}\tau_i} = A_i z_i + \gamma_i(y), \quad y_i = C_i z_i, \quad 1 \le i \le p, \tag{3}$$

where $z_i = (z_{i,1}, \cdots, z_{i,\lambda_i})^T$, $\gamma_i(y) = (\gamma_{i,1}(y), \cdots, \gamma_{i,\lambda_i}(y))^T$, and λ_i are the observability indices of system (1) [14]. The TOF for the entire system in time t is

$$\dot{z} = S(y)(Az + \gamma(y)), \quad y = Cz, \tag{4}$$

where $z = (z_1^T, \ldots, z_p^T)^T$, $\gamma(y) = (\gamma_1^T(y), \ldots, \gamma_p^T(y))^T$, and

$$S(y) = \text{Blockdiag}\{s_1(y)\mathbf{I}_{\lambda_1}, \ldots, s_p(y)\mathbf{I}_{\lambda_p}\}.$$

We remark that the difference between multi-output and single-output TOF is in the matrix $S(y)$. This difference leads to a different approach to derive the TOF existence conditions. Given TSF (2) and TOF (3), we have the following definition.

Definition 2.1: System (1) is said to be locally (globally) transformable to TOF (3) if there exists a local (global) diffeomorphism $z = \Phi(x)$ and time scaling transformations (2) such that the system can be represented as

$$\dot{z} = \frac{\partial \Phi(x)}{\partial x} f(x)\bigg|_{x=\Phi^{-1}(z)} = S(y)(Az + \gamma(y)),\ y = Cz. \tag{5}$$

Since TSFs are non-vanishing, we can multiply $S^{-1}(y)$ to both sides of (5) and obtain

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = Az + \gamma(y), \quad y = Cz,$$

where $\mathrm{d}z/\mathrm{d}\tau = (\mathrm{d}z_1/\mathrm{d}\tau_1, \cdots, \mathrm{d}z_p/\mathrm{d}\tau_p)^T$.

III. Existence Conditions

We first introduce some notation, then present the existence conditions for a TOF. Next, the necessary and sufficient conditions for a TOF where the same time scaling transformation is used for all subsystems are given; these conditions can be specified in a concise form and are similar to the established result for an OF. Following [4] we define two co-distributions Q_i, Q:

$$\begin{aligned} Q_i &= \text{span}\{\mathrm{d}L_f^k h_r(x), 0 \le k \le \lambda_i - 1, 1 \le r \le p\} \\ &\quad \backslash \{\mathrm{d}L_f^{\lambda_i - 1} h_i(x)\}, \quad 1 \le i \le p, \\ Q &= \text{span}\{\mathrm{d}L_f^k h_r(x), 0 \le k \le \lambda_r - 1, 1 \le r \le p\}. \end{aligned}$$

For system (1), it has been shown in [4, Thm. 2.3] that $Q_i = Q_i \cap Q$ guarantees the existence of the starting vector g_i satisfying

$$L_{g_i} L_f^k h_r(x) = \delta_{k,\lambda_i - 1}\delta_{i,r},\ 0 \le k \le \lambda_i - 1,\ 1 \le r \le p. \quad (6)$$

The starting vector g_i is generally not uniquely defined. When system (1) is in observable form [10], a typical solution of (6) is $g_i = \partial/\partial x_{i,\lambda_i}$.

A. Multiple Time Scaling Transformation Case

Theorem 3.1: System (1) is locally transformable to TOF (5) if and only if, locally for $1 \le i \le p$,

1) The TSF $s_i(y)$ of the ith subsystem satisfies the partial differential equations

$$\begin{aligned} \mathrm{d}L_{g_i} L_f^{\lambda_i} h_i(x) &= \frac{1}{s_i(y)} \Bigg(l_{\lambda_i} \frac{\partial s_i(y)}{\partial y_i} \mathrm{d}L_f h_i(x) \\ &+ (l_{\lambda_i} - 1) \sum_{j=1, j \ne i}^{p} \frac{\partial s_i(y)}{\partial y_j} \mathrm{d}L_f h_j(x) \Bigg) \bmod \{\mathrm{d}y\}, \end{aligned} \quad (7)$$

with $l_k = \frac{k(k-1)}{2} + 1,\ 1 \le k \le \lambda_i$, and g_i given by (6).

2) Q_i=$Q_i \cap Q$.

3) Given the starting vector fields $\bar{g}_i$

$$L_{\bar{g}_i} L_f^k h_l(x) = s_i^{\lambda_i - 1}(h(x)) \delta_{k,\lambda_i - 1}\delta_{l,i}, \quad (8)$$

for $0 \le k \le \lambda_i - 1; 1 \le l \le p$, and the vector fields $\eta_{i,1} = \bar{g}_i,\ \eta_{i,j} = s_i^{-1}(h(x))\mathrm{ad}_{-f}\eta_{i,j-1},\ 2 \le j \le \lambda_i$,

$$[\eta_{i,r}, \eta_{l,s}] = 0,\ 1 \le r \le \lambda_i; 1 \le s \le \lambda_l; 1 \le i, l \le p. \quad (9)$$

The transformation $z = \Phi(x)$ is the solution of the system of n^2 partial differential equations

$$\frac{\partial \Phi}{\partial x} \left[\eta_{1,\lambda_1}, \cdots, \eta_{1,1}, \cdots, \eta_{p,\lambda_p}, \cdots, \eta_{p,1} \right] = \mathbf{I}_n. \quad (10)$$

Remark 3.2: We express s_i in terms of y to emphasize its dependence on the output. The formula $s(h(x))$ is required to compute the starting vectors $\bar{g}_i$ and vector fields $\eta_{i,j}$, and verify Lie bracket conditions. We abbreviate $s_i(h(x))$ or $s_i(y)$ as s_i, $\gamma(y)$ as γ, and $S(y)$ as S, . Note that all the conditions of Theorem 3.1 are independent of the choice of coordinates. However, observable form coordinates are useful to simplify the calculation of s_i since g_i and $L_f h_i(x) = x_{i,2}$ have simple expressions in these coordinates.

Remark 3.3: The TOF coordinates are globally defined if system (1) is globally observable, the conditions in Theorem 3.1 are satisfied globally, and the vector fields $\eta_{i,j}, 1 \le j \le \lambda_i, 1 \le i \le p$ are complete. This is because the transformation $\Phi(x)$ can be constructed from the composition of flows of vector fields $\eta_{i,j}$, which is globally defined if the vector fields are complete.

Remark 3.4: Given system (1) in observable form, we know $y_i = x_{i,1}$, $L_f h_i = x_{i,2}$, and $L_f^{\lambda_i} h_i = \varphi^i(x)$ for $1 \le i \le p$. Taking $g_i = \partial/\partial x_{i,\lambda_i}$, we reformulate Condition (7) as

$$\begin{aligned} \frac{\partial^2 \varphi^i(x)}{\partial x_{i,2}\partial x_{i,\lambda_i}} &= \frac{l_{\lambda_i}}{s_i} \frac{\partial s_i}{\partial y_i}, \\ \frac{\partial^2 \varphi^i(x)}{\partial x_{k,2}\partial x_{i,\lambda_i}} &= \frac{l_{\lambda_i} - 1}{s_i} \frac{\partial s_i}{\partial y_k}, \quad 1 \le k \le p; k \ne i. \end{aligned}$$

Since $s_i > 0$, we introduce the change of variable $\kappa_i = \ln(s_i)$ and have

$$\frac{\partial \kappa_i}{\partial y_k} = \begin{cases} \frac{1}{l_{\lambda_i} - 1} \varphi^i_{k,i}(x), & k \ne i, \\ \frac{1}{l_{\lambda_i}} \varphi^i_{i,i}(x), & k = i, \end{cases}$$

where $\varphi^i_{k,i} = \partial^2 \varphi^i(x)/\partial x_{k,2}\partial x_{i,\lambda_i}$. A solution of κ_i exists if and only if $\frac{\partial^2 \kappa_i}{\partial y_j \partial y_k} = \frac{\partial^2 \kappa_i}{\partial y_k \partial y_j}$, which imposes conditions on $\varphi^i(x)$:

$$\begin{aligned} \frac{\partial \varphi^i_{k,i}(x)}{\partial y_j} &= \frac{\partial \varphi^i_{j,i}(x)}{\partial y_k}, \quad j \ne i, k \ne i, \\ \frac{1}{l_{\lambda_i}} \frac{\partial \varphi^i_{i,i}(x)}{\partial y_j} &= \frac{1}{l_{\lambda_i} - 1} \frac{\partial \varphi^i_{j,i}(x)}{\partial y_i}, \quad k = i, j \ne i. \end{aligned}$$

Remark 3.5: The necessary Condition (7) for s_i has the advantage of being relatively easy to present and verify. Indeed, as discussed in the previous remark, we can reexpress (7) into conditions involving only the system dynamics. Since Condition (7) is implied by Condition (9), we can expand Condition (9) to obtain necessary and sufficient conditions on s_i. The resulting differential equations would be more directly related to the system dynamics but would be complicated in general. A similar issue arises with feedback linearization in [11, Eqn. (22)] where a significant number of necessary conditions are given for the TSF. However, the solvability of these differential equations is difficult to discuss.

Remark 3.6: From Definition (8), we know $\bar{g}_i = s_i^{\lambda_i - 1} g_i$ and its existence is guaranteed by Condition 2) [4]. We note the following fact which is useful to show Theorem 3.1. Given (5), we can verify

$$\begin{aligned} &\mathrm{d}h_i(\Phi^{-1}(z)) \frac{\partial S(y)}{\partial y_i} (Az + \gamma(y)) \\ &= \mathrm{d}h_i(\Phi^{-1}(z)) \frac{\partial S(y)}{\partial y_i} S^{-1}(y) S(y)(Az + \gamma(y)) \\ &= \frac{\partial s_i(y)}{\partial y_i} \frac{1}{s_i(y)} L_f h_i(\Phi^{-1}(z)). \end{aligned}$$

Proof: $\Leftarrow$: The proof of necessity is to derive conditions of Theorem 3.1 in z-coordinates, which implies the conditions in the original coordinates since they are independent of coordinates. The computation is therefore carried out in z-coordinates. To simplify the notation, we denote $h(z)$ as the expression of $h(x)$ in z-coordinates instead of $h(\Phi^{-1}(z))$. Taking $\eta_{i,1} = \partial/\partial z_{i,\lambda_i}, 1 \le i \le p$ and following the definition of $\eta_{i,k}, 2 \le k \le \lambda_i$, we have $\eta_{i,k} = \partial/\partial z_{i,\lambda_i - k + 1}$. Clearly, $\eta_{i,k}, 1 \le k \le \lambda_i, 1 \le i \le p$ are unit vector fields in z-coordinates, i.e., Condition (9) is necessary. Next we derive the definition of the starting vector $\bar{g}_i$ (8). Since $\bar{g}_i = \eta_{i,1}, 1 \le i \le p$, we have $\partial h_l(z)/\partial z_{i,\lambda_i} = 0 \Rightarrow L_{\bar{g}_i} h_l(z) = 0$ for $1 \le l \le p$.

Further computation gives

$$0 = \frac{\partial h_l(z)}{\partial z_{i,\lambda_i - 1}} = \langle \mathrm{d}h_l(z), \eta_{i,2}\rangle$$
$$= \left\langle \mathrm{d}h_l(z), \frac{1}{s_i}[-f, \bar{g}_i]\right\rangle = \frac{1}{s_i} L_{\bar{g}_i} L_f h_l(z)$$

for $1 \le l \le p$. By induction, one can show for $1 \le l \le p$,

$$L_{\bar{g}_i} L_f^k h_l(z) = \begin{cases} s_i^k \frac{\partial h_l(z)}{\partial z_{i,\lambda_i - k}} = 0, \ 0 \le k \le \lambda_i - 2, \\ s_i^{\lambda_i - 1} \frac{\partial h_l(z)}{\partial z_{i,1}} = s_i^{\lambda_i - 1}, \ k = \lambda_i - 1. \end{cases}$$

Hence, the starting vector $\bar{g}_i$ satisfies (8). For the necessity of Condition 2), one can refer to [4] and the references therein.

To derive the condition on the TSFs, we first state the equations ensuring the existence of state transformation $\Phi(x)$, for $1 \le i \le p$

$$s_i \frac{\partial W}{\partial z_{i,j-1}} = \mathrm{ad}_{-f} \frac{\partial W}{\partial z_{i,j}}, \quad 2 \le j \le \lambda_i, \tag{11a}$$
$$\frac{\partial W}{\partial z} \frac{\partial}{\partial z_{i,1}} (S(Az + \gamma)) = \mathrm{ad}_{-f} \frac{\partial W}{\partial z_{i,1}}, \quad j = 1, \tag{11b}$$
$$\mathrm{d}h_r \frac{\partial W}{\partial z_{i,j}} = \delta_{j,1}\delta_{r,i}, \quad 1 \le j \le \lambda_i; 1 \le r \le p, \tag{11c}$$

where $W = \Phi^{-1}(z)$, and $\partial W/\partial z_{i,\lambda_i}$ is the starting vector $\bar{g}_i$. One can see from (11a) that $\partial W/\partial z_{i,j} = \eta_{i,\lambda_i - j + 1}, 1 \le j \le \lambda_i, 1 \le i \le p$.

The left hand side of (11b) is

$$\frac{\partial W}{\partial z} \frac{\partial (S(Az + \gamma))}{\partial z_{i,1}} = \frac{\partial W}{\partial z} \left(\frac{\partial S}{\partial z_{i,1}} (Az + \gamma) + S \frac{\partial \gamma}{\partial z_{i,1}} \right).$$

Given the right hand side of (11b) in Remark 3.7, (11b) multiplied by $\mathrm{d}h_i(z)(\partial W/\partial z)^{-1}$ is

$$\begin{aligned} &\mathrm{d}h_i(z) \frac{\partial S}{\partial z_{i,1}} (Az + \gamma) + \mathrm{d}h_i(z) S \frac{\partial \gamma}{\partial z_{i,1}} \\ &= \mathrm{d}h_i(z) \frac{\mathrm{ad}_{-f}^{\lambda_i} \bar{g}_i}{s_i^{\lambda_i - 1}} + \mathrm{d}h_i(z) \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i) \mathrm{ad}_{-f}^{\lambda_i - 1} \bar{g}_i. \end{aligned} \tag{12}$$

According to Remark 3.6, (12) is modified into

$$\begin{aligned} &\frac{\partial s_i}{\partial z_{i,1}} \frac{1}{s_i} L_f h_i(z) + \rho(y) \\ &= \frac{L_{\mathrm{ad}_{-f}^{\lambda_i} \bar{g}_i} h_i(z)}{s_i^{\lambda_i - 1}} + \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i) L_{\mathrm{ad}_{-f}^{\lambda_i - 1} \bar{g}_i} h_i(z). \end{aligned} \tag{13}$$

From [15, Lem. 4.1.2], [16, Thm. A.3.1], we have $L_{\mathrm{ad}_{-f}^{\lambda_i - 1} \bar{g}_i} h_i(z) = L_{\bar{g}_i} L_f^{\lambda_i - 1} h_i(z) = s_i^{\lambda_i - 1}$, and $L_{\mathrm{ad}_{-f}^{\lambda_i} \bar{g}_i} h_i(z) = L_{\bar{g}_i} L_f^{\lambda_i} h_i(z)$. Since $h_i(z) = h_i(x) = y_i$, (13) is rearranged as

$$\frac{\partial s_i}{\partial y_i} \frac{L_f h_i}{s_i} + \rho(y) = \frac{L_{\bar{g}_i} L_f^{\lambda_i} h_i}{s_i^{\lambda_i - 1}} - \frac{\lambda_i(\lambda_i - 1)}{2} \frac{1}{s_i} L_f(s_i). \tag{14}$$

Collecting the terms of (14) and taking the differential, we have

$$\begin{aligned} &\mathrm{d}L_{\bar{g}_i} L_f^{\lambda_i} h_i = l_{\lambda_i} s_i^{\lambda_i - 2} \frac{\partial s_i}{\partial y_i} \mathrm{d}L_f h_i \\ &+ (l_{\lambda_i} - 1) s_i^{\lambda_i - 2} \sum_{j=1, j \ne i}^{p} \frac{\partial s_i}{\partial y_j} \mathrm{d}L_f h_j \mod \{\mathrm{d}y\}. \end{aligned} \tag{15}$$

Since $\bar{g}_i = s_i^{\lambda_i - 1} g_i$, we have $\mathrm{d}L_{\bar{g}_i} L_f^{\lambda_i} h_i = s^{\lambda_i - 1} \mathrm{d}L_{g_i} L_f^{\lambda_i} h_i \mod \{\mathrm{d}y\}$. Hence, we have Condition (7) by plugging the above equation into (15).

$\Rightarrow$: Given the TSFs of each subsystem s_i solved from (7), it is readily shown conditions 2)–3) are sufficient to guarantee the existence of state coordinate $z = \Phi(x)$ which puts system (1) into TOF (4) by following the proof in [1], [4], [16]. ∎

Remark 3.7: Given (11a), one can compute $\partial W/\partial z_{i,j}, 1 \le j \le \lambda_i - 1$ iteratively and have

$$\begin{aligned} \frac{\partial W}{\partial z_{i,\lambda_i - k}} =& \frac{1}{s_i^k} \mathrm{ad}_{-f}^k \bar{g}_i + \frac{\sum_{j=1}^{k-1} j}{s_i^{k+1}} L_f(s_i) \mathrm{ad}_{-f}^{k-1} \bar{g}_i \\ &\mathrm{span}\ \{\mathrm{ad}_{-f}^j \bar{g}_i, 0 \le j \le k-2\}, 1 \le k \le \lambda_i - 1. \end{aligned}$$

Further calculation yields the right hand side of (11b)

$$\begin{aligned} \mathrm{ad}_{-f} \frac{\partial W}{\partial z_{i,1}} =& \frac{1}{s_i^{\lambda_i - 1}} \mathrm{ad}_{-f}^{\lambda_i} \bar{g}_i + \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i) \mathrm{ad}_{-f}^{\lambda_i - 1} \bar{g}_i \\ &\mathrm{span}\ \{\mathrm{ad}_{-f}^j \bar{g}_i, 0 \le j \le \lambda_i - 2\}. \end{aligned}$$

Remark 3.8: The multiple time scaling transformation case has a different TSF for each subsystem. This can be generalized by employing a TSF for each state, i.e.,

$$S(y) = \mathrm{Blockdiag}\{s_1(y), \ldots, s_n(y)\},$$

which leads to the multi-output extension of the output dependent observability linear normal form in [17]. Allowing distinct time scaling transformation for each state further enlarges the class of admissible systems transformable to OF. A similar procedure can be followed to obtain the existence conditions of the corresponding TOF.

B. Single Time Scaling Transformation Case

The existence conditions is given in the following theorem without proof.

Theorem 3.9: System (1) is locally transformable to TOF (5) if and only if, locally for $1 \le i \le p$

1) Condition 1) in Theorem 3.1 with $s(y) = s_i(y)$ holds.
2) $Q_i = Q_i \cap Q$.
3) the following Lie brackets conditions hold, i.e.,

$$\left[\mathrm{ad}_{-\bar{f}}^k \bar{g}_r, \mathrm{ad}_{-\bar{f}}^l \bar{g}_q\right] = 0, \quad \begin{cases} 0 \le k \le \lambda_r - 1; \\ 0 \le l \le \lambda_q - 1; \\ 1 \le r, q \le p, \end{cases} \tag{16}$$

where $\bar{f}(x) = f(x)/s(h(x))$, and $\bar{g}_i$ is the starting vector field in time τ and defined by

$$L_{\bar{g}_i} L_{\bar{f}}^k h_r = \delta_{k,\lambda_i - 1}\delta_{i,r}, \ 0 \le k \le \lambda_i - 1; 1 \le r \le p. \tag{17}$$

The transformation $z = \Phi(x)$ is the solution of the system of n^2 partial differential equations

$$\frac{\partial \Phi}{\partial x}\left[\mathrm{ad}_{-\bar{f}}^{\lambda_1 - 1} \bar{g}_1, \cdots, \bar{g}_1, \cdots, \mathrm{ad}_{-\bar{f}}^{\lambda_p - 1} \bar{g}_p, \cdots, \bar{g}_p\right] = \mathbf{I}_n. \tag{18}$$

Remark 3.10: The difference between Theorem 3.9 and [4, Thm. 3.4] is the additional Condition 1) on $s(y)$. Provided the existence of a TSF, the necessity and sufficiency of Conditions 2) – 3) have been shown in [4]. Condition 1) is a special case of Condition 1) in Theorem 3.1.

Remark 3.11: Assuming for system (1), the starting vectors $g_i, 1 \le i \le p$ can be solved from (6), we have the starting vectors defined by (17) $\bar{g}_i = s^{\lambda_i - 1} g_i$. This is because by induction, we derive that for a fixed i and any $r, 1 \le r \le p$,

$$\begin{aligned} \mathrm{d}L_{\bar{f}}^k h_r =& s^{-1}\mathrm{d}L_f h_r \mod \{\mathrm{d}h_r\}, \quad k=0, \\ \mathrm{d}L_{\bar{f}}^k h_r =& s^{-k}\mathrm{d}L_f^k h_r \mod \{\mathrm{d}L_f^j h_r, 0 \le j \le k-1\} \\ & 1 \le k \le \lambda_i - 1. \end{aligned}$$

Remark 3.12: The multiple time scaling transformations are a generalization of the single time scaling transformation case. If we replace $S(y)$ with $s(y)$, Theorem 3.1 is equivalent to Theorem 3.9. We can verify the $\bar{g}_i$ solved from (8) is the same as $\bar{g}_i$ solved from (17), and $\mathrm{ad}_{-\bar{f}}^{k-1}\bar{g}_i = \eta_{i,k}, 1 \le k \le \lambda_i, 1 \le i \le p$. Thus the Lie bracket conditions are equivalent. When $p=1$, Theorem 3.9 and Theorem 3.1 lead to the same existence conditions as [8, Thm. 1].

C. OF, TOF, and OF with Output Transformation

We discuss differences between TOF and OF with output transformation [3]. Our discussion relies on a system (1) being in observable form with indices $\lambda_i, 1 \le i \le p$. Given $g_i = \partial/\partial x_{i,\lambda_i}$ and $L_f^{\lambda_i} h_i = \varphi_i(x)$, Condition (7) is

$$\begin{aligned} \mathrm{d}\frac{\partial \varphi_i(x)}{\partial x_{i,\lambda_i}} =& \frac{1}{s_i}\left(l_{\lambda_i}\frac{\partial s_i}{\partial y_i}\mathrm{d}x_{i,2} + (l_{\lambda_i} - 1)\sum_{j=1, j\neq i}^{p} \mathrm{d}x_{j,2}\right) \\ & \mod \{\mathrm{d}y\}, \quad 1 \le i \le p. \end{aligned}$$

Performing coefficient matching of the above equation, we can solve s_i only if φ_i is affine in x_{i,λ_i} and the coefficients of x_{i,λ_i} in φ_i are of the form $\alpha_1(y)x_{j,2}$ or $\alpha_2(y)$. However, without the time scale transformation, the necessary condition for OF requires no terms of the form $\alpha_1(y)x_{j,2}x_{i,\lambda_i}$ in φ_i. This illustrates a benefit of introducing a time scaling transformation. If $\lambda_i = 1$ and $\varphi_i(x)$ has dependence on $x_{j,k}, k \ge 2$, no TOF can be solved. On the other hand, if φ_i has linear dependence on $x_{j,2}$, introducing the output transformation leads to the transformability to OF.

We perform the comparison between a time scaling transformation and an output transformation by considering a p-output system with observability indices $\lambda_k = 2, 1 \le k \le p$. Assuming the system is in observable form, $\bar{g}_i, 1 \le i \le p$ are unit vectors and therefore commute. We check $[\eta_{i,2}, \eta_{k,1}] = \left[s_i^{-1}\mathrm{ad}_{-f}\bar{g}_i, \bar{g}_k\right] = 0$ to derive necessary conditions for a TOF

$$\begin{aligned} &\left[s_i^{-1}\mathrm{ad}_{-f}\bar{g}_i, \bar{g}_k\right] = \left[\mathrm{ad}_{-f}g_i - s_i^{-1}L_f(s_i)g_i, s_k g_k\right] \\ &= s_k\left[\mathrm{ad}_{-f}g_i, g_k\right] + L_{\mathrm{ad}_{-f}g_i}(s_k)g_k + \left[-s_i^{-1}L_f(s_i)g_i, s_k g_k\right] \\ &= s_k \sum_{l=1}^{p} \frac{\partial^2 \varphi_l}{\partial \dot{y}_i \partial \dot{y}_k}\frac{\partial}{\partial \dot{y}_l} + \frac{\partial s_k}{\partial y_i}\frac{\partial}{\partial \dot{y}_k} + \frac{s_k}{s_i}\frac{\partial s_i}{\partial y_k}\frac{\partial}{\partial \dot{y}_i}, \end{aligned}$$

which yields the partial differential equations

$$\frac{\partial^2 \varphi_i}{\partial \dot{y}_i^2} + \frac{2}{s_i}\frac{\partial s_i}{\partial y_i} = 0, \tag{19a}$$

$$\frac{\partial^2 \varphi_k}{\partial \dot{y}_i \partial \dot{y}_k} + \frac{1}{s_k}\frac{\partial s_k}{\partial y_i} = 0, \quad k \neq i, \tag{19b}$$

$$\frac{\partial^2 \varphi_l}{\partial \dot{y}_i \partial \dot{y}_k} = 0, \quad l \neq k; l \neq i, \tag{19c}$$

for $1 \le l, i, k \le p$. For the output transformation case, we define the output transformation $\bar{y} = \psi(y) = (\psi_1, \cdots, \psi_p)^T$ and compute $L_f^2 \bar{y}_i = \sum_{l=1}^{p} \frac{\partial \psi_i}{\partial y_l}\varphi^l + \sum_{k=1}^{p}\sum_{i=1}^{p} \frac{\partial^2 \psi_i}{\partial y_k \partial y_i}\dot{y}_i \dot{y}_k$. Transformability to OF with output transformation requires

$$\begin{aligned} \sum_{l=1}^{p} \frac{\partial \psi_i}{\partial y_l}\frac{\partial^2 \varphi_l}{\partial \dot{y}_k \partial \dot{y}_i} + \frac{\partial^2 \psi_i}{\partial y_i \partial y_k} = 0, \quad i \neq k, \\ \sum_{l=1}^{p} \frac{\partial \psi_i}{\partial y_l}\frac{\partial^2 \varphi_l}{\partial \dot{y}_i^2} + 2\frac{\partial^2 \psi_i}{\partial y_i^2} = 0, \quad i = k, \end{aligned} \tag{20}$$

for $1 \le i, k, l \le p$. Comparing Conditions (19) and (20), we recover the result in [8] that an output transformation is equivalent to a time scale transformation. This is because when $p=1, n=2, i=k=l$, (19) and (20) are not only sufficient but also equivalent. We next present examples, which violate either (19) or (20), to show that an output transformation is in general not equivalent to a time scale transformation.

Consider the example system with indices $(2,2,2)$ which does not satisfy (19c), i.e., no TOF exists, but it is transformable to OF with output transformation:

$$\begin{aligned} \dot{x}_1 &= (x_{12} + \gamma_{11}(y_2))\frac{\partial}{\partial x_{11}} + \gamma_{12}(y)\frac{\partial}{\partial x_{12}}, \quad y_1 = x_{11}, \\ \dot{x}_2 &= (x_{22} + \gamma_{21}(y_1))\frac{\partial}{\partial x_{21}} + \gamma_{22}(y)\frac{\partial}{\partial x_{22}}, \quad y_2 = x_{21}, \\ \dot{x}_3 &= x_{32}\frac{\partial}{\partial x_{31}} + (x_{12}x_{22} + x_{32})\frac{\partial}{\partial x_{32}}, \quad y_3 = x_{31}, \end{aligned}$$

where $x_1 = (x_{11}, x_{12})^T, x_2 = (x_{21}, x_{22})^T, x_3 = (x_{31}, x_{32})^T$. This system is not transformable to OF without output transformation since $x_{12}x_{22}$ appears in $\varphi_3 = \dot{x}_{32}$. Solving for the output transformation $\psi_3 = y_1 y_2 - 2y_3$, the system with new output $y = (x_{11}, x_{21}, \psi_3)^T$ is transformable to OF.

Consider a system in observable form with indices $(2,2)$

$$\begin{aligned} \dot{x}_1 =& x_{12}\frac{\partial}{\partial x_{11}} + (x_{11}x_{12}x_{22} + x_{21}x_{12}^2)\frac{\partial}{\partial x_{12}}, \ y_1 = x_{11}, \\ \dot{x}_2 =& x_{22}\frac{\partial}{\partial x_{21}} + (x_{12}x_{21}x_{22} + x_{11}x_{22}^2)\frac{\partial}{\partial x_{22}}, \ y_2 = x_{21}. \end{aligned} \tag{21}$$

System (21) is not transformable to OF since $x_{12}^2, x_{12}x_{22}$, and x_{22}^2 are present in φ_1 and φ_2, respectively. We apply Theorem 3.1 to investigate whether system (21) admits a TOF. The starting vectors are $g_1 = \partial/\partial x_{12}, g_2 = \partial/\partial x_{22}$ and we have

$$\begin{aligned} L_{g_1}L_f^2 h_1 = x_{11}x_{22} + 2x_{21}x_{12}, \\ L_{g_2}L_f^2 h_2 = x_{21}x_{12} + 2x_{11}x_{22}. \end{aligned}$$

With $L_f h_k = x_{k2}, k = 1,2$, Condition (7) in Theorem 3.1 yields partial differential equations

$$\begin{aligned} 2y_2 = \frac{1}{s_1}\frac{\partial s_1}{\partial y_1}, \quad y_1 = \frac{1}{s_1}\frac{\partial s_1}{\partial y_2}, \\ 2y_1 = \frac{1}{s_2}\frac{\partial s_2}{\partial y_2}, \quad y_2 = \frac{1}{s_2}\frac{\partial s_2}{\partial y_1}. \end{aligned}$$

Solving these equations gives $s_1 = s_2 = e^{y_2 y_1}$. With $s_1 = s_2$, Theorem 3.9 can be applied. Defining $\bar{f} = f/s_1, \bar{g}_k = s_1 g_k, k = 1,2$, we verify Condition (16) and conclude that system (21) is transformable to a TOF. On the other hand, system (21) cannot be put into an OF with an output transformation since no output transformation satisfies (20).

D. Observer Design, Implementation and Robustness Issues

Assuming the existence of a TOF, a Kalman-like observer design can be performed as in [18, Thm. 3.1]. The resulting error dynamics is guaranteed to be exponentially stable. However, using this design would require additional observer states to compute a time-varying observer gain by numerical integration. We propose a relatively simple Luenberger observer in TOF coordinates

$$\dot{\hat{z}} = S(y)(A\hat{z} + \gamma(y) + L(y - C\hat{z})). \tag{22}$$

This proposed observer yields the error dynamics

$$\dot{\tilde{z}} = S(y)(A - LC)\tilde{z}. \tag{23}$$

We have the following result on the stability of (23).

Theorem 3.13: Provided that system (1) is globally transformed to TOF (3) and given the observer (22) with $A - LC$ Hurwitz, the zero solution of the error dynamics (23) is uniformly globally exponentially stable if and only if there exist positive constants ϵ_1, ϵ_2, such that

$$\int_0^t s_i(y(\xi))\mathrm{d}\xi \geq \epsilon_1 t + \epsilon_2, \quad \forall t \geq 0. \tag{24}$$

Relative to the result in [18] an additional condition (24) is required to ensure the error dynamics stability. However, the proposed observer benefits from a simpler observer gain and implementation.

Proof: We only need to prove the stability of

$$\dot{\tilde{z}}_i = s_i(y(t))(A_i - L_iC_i)\tilde{z}_i,\ t \geq 0.$$

$\Leftarrow$: We assume the zero solution of error dynamics is UGES in both τ and t times, i.e.

$$b_1 e^{-b_2\tau} \leq \|\tilde{z}_i(\tau)\| \leq c_1 e^{-c_2\tau}, \tag{25a}$$

$$b_3 e^{-b_4 t} \leq \|\tilde{z}_i(t)\| \leq c_3 e^{-c_4 t}, \tag{25b}$$

where $c_k, b_k, 1 \leq k \leq 4$ are positive constants. Substituting $\tau(t) = \int_0^t s_i(y(\xi))\mathrm{d}\xi$ into (25a), we have

$$b_1 e^{-b_2\int_0^t s_i(y(\xi))\mathrm{d}\xi} \leq \|\tilde{z}_i \circ \tau(t)\| \leq c_1 e^{-c_2\int_0^t s_i(y(\xi))\mathrm{d}\xi}.$$

Combining the last inequality with (25b) gives

$$b_1 e^{-b_2\int_0^t s_i(y(\xi))\mathrm{d}\xi} \leq c_3 e^{-c_4 t}$$

which implies condition (24) with $\epsilon_1 \geq c_4/b_2, \epsilon_2 \geq \frac{1}{b_2}\ln\frac{b_1}{c_3}$.

$\Rightarrow$: Since the zero solution of $\tilde{z}_i$ in τ time is UGES and condition (24) holds, we assume $\|\tilde{z}_i(\tau)\| \leq c_1 e^{-c_2\tau}$ and readily conclude that the zero solution of (23) in t time is UGAS. This is because

$$\|\tilde{z}_i \circ \tau(t)\| \leq c_3 e^{-c_4 t},$$

with $c_3 \geq c_1 e^{-c_2\epsilon_2}, c_4 \geq c_2\epsilon_1$. ■

Introducing multiple time scales does not lead to any difficulty in the implementation of the observer. This can be seen by expressing (22) in x-coordinates and time t. For simplification, assuming $S\partial\Phi(x)/\partial x = \partial\Phi(x)/\partial x S$, we have

$$\begin{aligned}\dot{\hat{x}} &= \frac{\partial\hat{x}}{\partial\hat{z}}S(y)(A\hat{z} + \gamma(y) + L(y - C\hat{z}))\\ &= S(y)S^{-1}(\hat{y})\frac{\partial\hat{x}}{\partial\hat{z}}S(\hat{y})(A\hat{z} + \gamma(y) + L(y - C\hat{z}))\\ &= S(y)\left(S^{-1}(\hat{y})f(\hat{x}) + \frac{\partial\hat{x}}{\partial\hat{z}}(\gamma(y) - \gamma(\hat{y}) + L(y - \hat{y}))\right).\end{aligned} \tag{26}$$

We consider the robustness of the error dynamics to measurement noise in z-coordinates and time t. With the measurement noise, e.g. $y_w = y + w(t)$, the observer is

$$\dot{\hat{z}} = S(y_w)(A\hat{z} + \gamma(y_w) + L(y_w - C\hat{z})),$$

and the corresponding error dynamics is

$$\begin{aligned}\dot{\tilde{z}} =& S(y_w)(A - LC)\tilde{z} + (S(y) - S(y_w))Az\\ &+ S(y)\gamma(y) - S(y_w)(\gamma(y_w) + Lw(t)).\end{aligned} \tag{27}$$

For simplicity, we assume that the zero solution of $\dot{\tilde{z}} = S(y_w)(A - LC)\tilde{z}$ is globally exponentially stable, and conclude that the solution of (27) evolves in a bounded set. This is a similar situation to the Luenberger observer based on an OF which also provides state estimate with bounded errors in the face of measurement noise.

IV. Numerical Examples

A. Single Time Scaling Transformation Example

We consider a two-output system in observable form with observability indices $(2, 2)$ corresponding to the output $y = (y_1, y_2)^T$.

$$\begin{aligned}\dot{x}_1 &= x_{12}\frac{\partial}{\partial x_{11}} + (x_{12}^2 + x_{12}x_{22})\frac{\partial}{\partial x_{12}}, \quad y_1 = x_{11},\\ \dot{x}_2 &= x_{22}\frac{\partial}{\partial x_{21}} + (x_{22}^2 + x_{12}x_{22})\frac{\partial}{\partial x_{22}}, \quad y_2 = x_{21},\end{aligned} \tag{28}$$

where $x_1 = (x_{11}, x_{12})^T, x_2 = (x_{21}, x_{22})^T$. We can verify system (28) is not transformable to OF by a change of state coordinates since the Lie bracket condition in [16, Thm. 5.4.1] is violated. Next, we apply Theorem 3.9 to solve the scalar TSF, verify the conditions, and compute the state transformation. We first solve g_i from (6) $g_1 = \partial/\partial x_{12},\ g_2 = \partial/\partial x_{22}$, and compute $L_{g_i}L_f^2 h_i = 2x_{i2} + x_{12}x_{22}, \mathrm{d}L_f h_i = x_{i2}, i = 1, 2$. Condition (7) is reduced to

$$\mathrm{d}(2x_{12} + x_{22}) = s^{-1}(2\frac{\partial s}{\partial y_1}\mathrm{d}x_{12} + \frac{\partial s}{\partial y_2}\mathrm{d}x_{22}) \mod \{\mathrm{d}y\},$$

$$\mathrm{d}(2x_{22} + x_{12}) = s^{-1}(2\frac{\partial s}{\partial y_2}\mathrm{d}x_{22} + \frac{\partial s}{\partial y_1}\mathrm{d}x_{12}) \mod \{\mathrm{d}y\}.$$

We therefore set up the partial differential equations $\frac{\partial s}{\partial y_k} = s, k = 1, 2$ and solve the scalar TSF $s(y) = e^{y_1 + y_2}$. With the starting vector fields $\bar{g}_k = sg_k,\ k = 1, 2$, Lie bracket conditions (16) hold for $0 \leq k, l \leq 1; 1 \leq r, q \leq 2$. The state transformation is computed as $\Phi(x) = (x_{11}, x_{12}/s, x_{21}, x_{22}/s)^T$ by solving (18).

B. Multiple Time Scaling Transformations Example

If we modify the dynamics of system (28) by taking

$$\dot{x}_{12} = x_{12}^2 + x_{21} - 2x_{12} - x_{11},$$

$$\dot{x}_{22} = x_{22}^2 + x_{11} - 2x_{22} - x_{21},$$

Theorem 3.1 can be applied to solve the matrix TSF, verify the conditions, and compute the transformation. Using (6) we solve the same starting vector fields $g_i, i = 1, 2$ as in the previous example and have $L_{g_i}L_f^2h_i = 2x_{i2}, L_fh_i(x) = x_{i2}, i = 1, 2$. Condition (7) is reduced to

$$2\mathrm{d}x_{12} = s_1^{-1}(2\frac{\partial s_1}{\partial y_1}\mathrm{d}x_{12} + \frac{\partial s_1}{\partial y_2}\mathrm{d}x_{22}) \mod \{\mathrm{d}y\},$$
$$2\mathrm{d}x_{22} = s_2^{-1}(2\frac{\partial s_2}{\partial y_2}\mathrm{d}x_{22} + \frac{\partial s_2}{\partial y_1}\mathrm{d}x_{12}) \mod \{\mathrm{d}y\},$$

which yields the partial differential equations

$$\frac{\partial s_1}{\partial y_1} = s_1, \quad \frac{\partial s_1}{\partial y_2} = 0, \quad \frac{\partial s_2}{\partial y_1} = 0, \quad \frac{\partial s_2}{\partial y_2} = s_2.$$

Hence, we solve the TSFs $s_1 = e^{y_1}, s_2 = e^{y_2}$ and verify Lie bracket conditions (9) for $1 \le r, s, i, l \le 2$. The change of coordinates is solved as $\Phi(x) = (x_{11}, (x_{12}-2)/s_1, x_{21}, (x_{22}-2)/s_2)^T$, which transforms the system in the form of (5) with $\gamma(y) = (2e^{-y_1}, e^{-2y_1}(y_2 - y_1), 2e^{-y_2}, e^{-2y_2}(y_1 - y_2))^T$. We implement the observer (22) in the original state coordinates and time as (26). To ensure TSFs satisfy Condition (24), we avoid $y \to \infty$. Hence, we choose the initial condition of the original system to ensure the boundedness of the trajectory $x(t)$. Taking the initial condition and observer gain as $x(0) = (-0.4, 1, -0.7, -0.5)^T$, $\hat{x} = 0, L = (l_1, l_2)$ with $l_1 = (4, 4, 0, 0)^T, l_2 = (0, 0, 4, 4)^T$, we have the simulation result in Figure 1. Simulation demonstrates local error dynamics stability and the ease of implementation of the observer using multiple time scales.

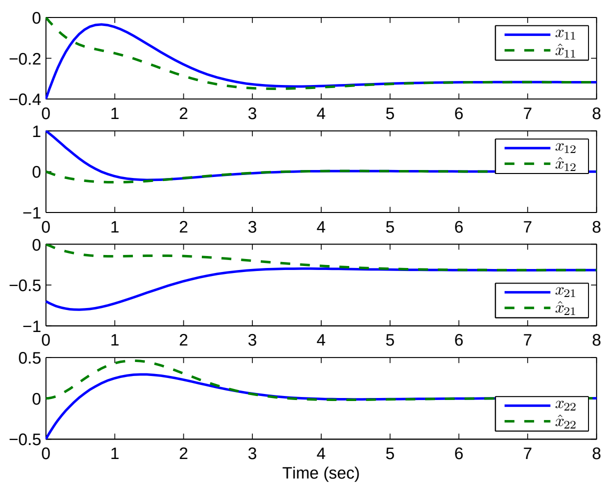

Fig. 1. Actual and estimated states of the system

V. Conclusion

Time scaling of the multi-output observer form for uncontrolled nonlinear continuous-time systems is considered in this note. Necessary and sufficient existence conditions of a time-scaled observer form are provided. Numerical examples show the construction of the state and time scaling transformations, and the implementation of an observer with multiple time scaling transformations.

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