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TR2015-145 December 2015

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2015 IEEE Conference on Decision and Control (CDC)

Reduced Complexity Control Design for Symmetric LPV Systems

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I. Introduction

Linear parameter varying (LPV) systems have been used in multiple contexts, such as gain scheduling, robust control, and control of periodic systems [1]. The recent survey [2] discusses the impact of several methods for controlling LPV systems with applications in different domains. A subclass of LPV systems are polytopic LPV systems [3], [4]. In polytopic LPV systems, the dynamics belong to the convex-hull of a finite set of extreme-dynamics, often called the *extreme systems*. The trajectories of a polytopic LPV system can also be obtained as trajectories of a polytopic difference inclusion with the same extreme systems [5].

A method for designing parameter-dependent controllers for discrete-time polytopic LPV systems based on polytopic parameter-dependent Lyapunov was presented in [6]. Parameter-dependent Lyapunov functions (pLF) were first introduced in [7] as a tool for robust stability analysis of uncertain autonomous systems. A pLFs is the convex combination of Lyapunov functions for the extreme dynamics. Stability is guaranteed by ensuring that the parameter-dependent Lyapunov function value decreases regardless of the evolution of the LPV parameter. Parameter-dependent Lyapunov functions are a less conservative alternative to using a common Lyapunov function for all the extreme systems. By considering functions that depend on the system parameter, pLFs provide less conservative stability certification.

In [7] it was shown how pLFs could be used for designing a robust linear controller for uncertain non-autonomous systems modeled as polytopic LPVs. As opposed to [7], in [6] the parameter of the LPV system at the current time is assumed to be known. A parameter-dependent controller can be designed, where the controller gains depends on the current LPV system parameter. Hence the controller gain adjusts as the LPV dynamics change. A parameter-dependent Lyapunov function is used to certify the stability of the LPV system in closed-loop with the parameter-dependent controller. More recently in [8] this method was extended to design output feedback controllers, where both the estimator and the controller are designed based on parameter-dependent Lyapunov functions. The methods in [7] and [6] have been extended to model predictive control (MPC), respectively for robust MPC [9] and MPC of LPV systems [10].

The design methods for parameter-dependent controllers and Lyapunov functions presented in [6]–[10] require the solution of linear matrix inequalities (LMIs) [5]. The size of the LMIs is proportional to the dimension of the system times the number of extreme systems. The multiple depends on the specific design method and objectives. The LMIs can become very large when the pLFs and parameter-dependent controller are designed simultaneously for LPV systems with many extreme system. Thus designing parameter-dependent controllers and Lyapunov functions can be a challenging problem. This is especially true in applications to MPC [9], [10], where the LMI are solved in real time during controller execution.

In this paper we propose a method for reducing the number and dimension of the design variables and the number of linear matrix inequalities using symmetry. For linear parameter varying systems, symmetries are transformations of the states and inputs that do not change the set of systems. Symmetries relate the extreme-dynamics of the LPV systems. These relationships can be exploited to reduce the number of extreme-controllers and extreme-Lyapunov functions that must be designed. In addition these symmetric relationships can be used to reduce the number of linear matrix inequalities that the design variables must satisfy.

Symmetry has a long history of being used to analyze and simplify control design [11]. In [12] it was shown that controllability and stability in large scale systems can be determined by using symmetry to decompose a system and then check the smaller decoupled subsystems. In [13] symmetry adapted basis was used to simplify the design of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers. In [14] symmetry was used to simplify the process of finding the transition probabilities of Markov-Chains which produce the fastest convergence. In [15] symmetry was used to simplify explicit model predictive controllers.

This paper is organized as follows. In Section II we define linear parameter varying systems, and parameter-dependent controllers and Lyapunov functions. We summarize a linear matrix inequality based procedure for designing parameter-dependent controllers and Lyapunov functions. In Section III we define symmetry for linear parameter varying systems, and parameter-dependent controllers and Lyapunov functions. In Section IV we provide three complementary methods for reducing the complexity of the controller and Lyapunov function design. In Section V we apply our reduced complexity control design to building control. We show that our method reduces the number of decision variables and linear matrix inequalities from a combinatorial function of the number of rooms to a polynomial function. Proofs have been omitted for space.

A. Mathematical Background on Group Theory

In this section we review the relevant concepts, notation, and results from group theory [16], [17]. Permutations will be described using the Cayley notation i.e. the permutation π such that $\pi(1) = 2$, $\pi(2) = 3$, and $\pi(3) = 1$ will be denoted as $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$.

A group $(\mathcal{G}, \circ)$ is a set $\mathcal{G}$ along with a binary operator $\circ$ such that the operator is associative, the set $\mathcal{G}$ is closed under the operation $\circ$, the set includes an identity element and the inverse of each element. In matrix groups the operator $\circ$ is matrix multiplication and the identity element is the identity matrix I. In permutation groups

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the operator $\circ$ is function composition and the identity element is the identity permutation e. A group that contains only the identity element is called trivial. For notational simplicity we will drop the $\circ$ and write gh for $g \circ h$.

A group $\mathcal{G}$ acts on a set $\mathcal{X}$ if its elements $g \in \mathcal{G}$ index functions $f_g : \mathcal{X} \to \mathcal{X}$ satisfying the group law: $f_g \circ f_h = f_{gh}$ where $\circ$ is function composition. A matrix group is defined in terms of its action $\Theta_g \in \mathbb{R}^{n\times n}$ on a vector-space $\mathbb{R}^n$ and a permutation group is defined by its action π_g on an index set $\mathcal{I} \subset \mathbb{N}$.

The orbit $\mathcal{G}(i) = \{\pi_g(i) : g \in \mathcal{G}\} \subseteq \mathcal{I}$ is the image $\pi_g(i)$ of the index $i \in \mathcal{I}$ under every permutation π_g for $g \in \mathcal{G}$ in the group $\mathcal{G}$. We will denote the set of orbits $\mathcal{G}(i)$ by $\mathcal{I}/\mathcal{G} = \{\mathcal{G}(i_1), \dots, \mathcal{G}(i_r)\}$, read as $\mathcal{I}$ modulo $\mathcal{G}$, where $i_1, \dots, i_r$ is a set that contains one representative i_j from each orbit $\mathcal{G}(i_j) \subseteq \mathcal{I}$. With abuse of notation we will equate the set of orbits $\mathcal{I}/\mathcal{G}$ with a set of representative indices $\mathcal{I}/\mathcal{G} = \{i_1, \dots, i_r\}$. The orbits $\mathcal{G}(i)$ for $i \in \mathcal{I}/\mathcal{G}$ partition the index set $\mathcal{I}$ into disjoint equivalence classes $\mathcal{I} = \bigsqcup_{i\in\mathcal{I}/\mathcal{G}} \mathcal{G}(i)$.

The stabilizer subgroup $\mathcal{G}_i = \{g \in \mathcal{G} : \pi_g(i) = i\}$ is the subset of group elements $g \in \mathcal{G}$ that map $i \in \mathcal{I}$ to itself $\pi_g(i) = i$. The size of the orbit $\mathcal{G}(i)$ and stabilizer $\mathcal{G}_i$ are related by the Orbit-Stabilizer theorem which states $|\mathcal{G}| = |\mathcal{G}(i)||\mathcal{G}_i|$ where $|\cdot|$ denotes the cardinality of a set.

The orbitals $\mathcal{G}(i,j) = \{(\pi_g(i), \pi(j)) \in \mathcal{I}^2 : g \in \mathcal{G}\} \subseteq \mathcal{I}^2$ are orbits of pairs $(i,j) \in \mathcal{I}^2$ of indices $i,j \in \mathcal{I}$. We denote the set of orbitals by $\mathcal{I}^2/\mathcal{G} = \{\mathcal{G}(i_1,j_1), \dots, \mathcal{G}(i_r,j_r)\}$ read as $\mathcal{I}\times\mathcal{I}$ modulo $\mathcal{G}$ where $\{(i_1,j_1), \dots, (i_r,j_r)\}$ is a set that contains one representative pair $(i_k,j_k) \in \mathcal{I}^2$ from each orbital $\mathcal{G}(i_k,j_k) \subseteq \mathcal{I}^2$. With a small abuse of notation we equate the set of orbitals $\mathcal{I}^2/\mathcal{G}$ with a set of representative pairs $\mathcal{I}^2/\mathcal{G} = \{(i_1,j_1), \dots, (i_r,j_r)\}$. The orbitals $\mathcal{G}(i,j)$ partition the set $\mathcal{I}^2$ into disjoint equivalence classes $\mathcal{I}^2 = \bigsqcup_{(i,j)\in\mathcal{I}^2/\mathcal{G}} \mathcal{G}(i,j)$.

Let $\Theta_g \in \mathbb{R}^{n\times n}$ and $\Omega_g \in \mathbb{R}^{m\times m}$ be matrix representations of the group $\mathcal{G}$. The commutator $\mathcal{G}(\Theta,\Omega) = \{M \in \mathbb{R}^{n\times m} : \Theta_g M = M\Omega_g, \forall g \in \mathcal{G}\}$ is the set of all matrices $M \in \mathbb{R}^{n\times m}$ that commute $\Theta_g M = M\Omega_g$ with Θ_g and Ω_g for all $g \in \mathcal{G}$. The commutator $\mathcal{G}(\Theta,\Omega) \subseteq \mathbb{R}^{n\times m}$ is a subspace of the vector-space $\mathbb{R}^{n\times m}$. This subspace $\mathcal{G}(\Theta,\Omega)$ can be described as the span of basis-matrices $\mathrm{M}_i \in \mathbb{R}^{n\times m}$ for $i = 1, \dots, d$ where $d \le nm$ is the dimension of the vector-space $\mathcal{G}(\Theta,\Omega)$. These basis-matrices can be found using techniques from representation theory [17], [18].

II. Linear Parameter Varying Systems

In this section we define linear parameter varying (LPV) systems, parameter-dependent controllers, and parameter-dependent Lyapunov functions. We review a procedure for designing parameter-dependent controllers and Lyapunov functions presented in the literature [6], [8]–[10], [19].

A. Linear Parameter Varying Systems

In this paper we study the following polytopic linear parameter varying system

$$x(t+1) = A(\xi)x(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and the parameter varying state-update matrix $A(\xi) = \sum_{i=1}^v \xi_i A_i$ lies inside the polytope $\mathcal{A} = \{\sum_{i=1}^v \xi_i A_i : \xi \ge 0, \sum_{i=1}^v \xi_i = 1\}$. We assume this is the minimal representation of $\mathcal{A}$. The matrices $A_i \in \mathbb{R}^{n\times n}$ are called the extreme state-update matrices of $\mathcal{A} \subset \mathbb{R}^{n\times n}$ are indexed by $\mathcal{I} = \{1, \dots, v\}$. We assume the time-varying parameter ξ is known at the current time t and lies in the simplex $\Xi = \{\xi \in \mathbb{R}^v : \xi \ge 0, \sum_{i=1}^v \xi_i = 1\}$.

Our objective is to design a controller that stabilizes the system (1) for any value of the time-varying parameter ξ. We use a parameter-dependent control law

$$u(t) = K(\xi)x(t) \tag{2}$$

where the parameter varying controller gain $K(\xi) = \sum_{i\in\mathcal{I}} \xi_i K_i \in \mathbb{R}^{m\times n}$ lies inside the polytope $\mathcal{K} = \{\sum_{i\in\mathcal{I}} \xi_i K_i : \xi \in \Xi\}$. The gains $K_i \in \mathbb{R}^{m\times n}$ are called the extreme controller gains of $\mathcal{K} \subset \mathbb{R}^{m\times n}$ are indexed by $\mathcal{I}$.

The stability of the LPV system (1) in closed-loop with the parameter-dependent controller (2) can be certified using a parameter-dependent Lyapunov function

$$V(x,\xi) = x^\intercal P(\xi) x \tag{3}$$

where there exists scalars $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x\|^2 \le x^\intercal P(\xi) x \le \alpha_2 \|x\|^2 \tag{4}$$

for all $\xi \in \Xi$. The parameter varying Lyapunov matrix $P(\xi) = \sum_{i\in\mathcal{I}} \xi_i P_i \in \mathbb{R}^{n\times n}$ lies inside the polytope $\mathcal{P} = \{\sum_{i\in\mathcal{I}} \xi_i P_i : \xi \in \Xi\}$. The matrices $P_i = P_i^\intercal \succ 0 \in \mathbb{R}^{n\times n}$ are called the extreme Lyapunov matrices of $\mathcal{P} \subset \mathbb{R}^{n\times n}$ are indexed by $\mathcal{I}$. The closed-loop system is asymptotically stable if

$$\big(A(\xi) + BK(\xi)\big)^\intercal P(\xi^+)\big(A(\xi) + BK(\xi)\big) - P(\xi) \prec 0 \tag{5}$$

for all $\xi, \xi^+ \in \Xi$ where $\xi^+ = \xi(t+1)$. This condition ensures $V(x(t+1), \xi(t+1)) - V(x(t), \xi(t)) < 0$ for all $x(t) \ne 0$ and $\xi(t), \xi(t+1) \in \Xi$.

B. Design of parameter-dependent Controllers and Lyapunov Functions

In this section we review the process presented in [6], [9], [10], [19] for synthesizing parameter-dependent controllers and Lyapunov functions.

The extreme controller gains $K_i = E_i G_i^{-1}$ are parameterized by design variables $E_i \in \mathbb{R}^{m\times n}$ and $G_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I}$. The extreme Lyapunov matrices $P_i = S_i^{-1}$ are parameterized by design variables $S_i = S_i^\intercal \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I}$. The design variables E_i, G_i, and S_i are obtained by solving the following linear matrix inequalities

$$\underbrace{\begin{bmatrix} G_i + G_i^\intercal - S_i & (A_iG_i + BE_i)^\intercal & G_i^\intercal & E_i^\intercal \\ (A_iG_i + BE_i) & S_j & 0 & 0 \\ G_i & 0 & Q^{-1} & 0 \\ E_i & 0 & 0 & R^{-1} \end{bmatrix}}_{\mathrm{LMI}(G_i,E_i,S_i,S_j)} \succeq 0 \tag{6}$$

for $i,j \in \mathcal{I}$ where $\mathrm{LMI}(G_i,E_i,S_i,S_j)$ is short-hand for the compound matrix above. The matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are design variables, similar to the penalty matrices in LQR, used to shape the controller performance. In [10], [19] it was shown that the parameter-dependent controller $K_i = E_i G_i^{-1}$ and Lyapunov function $P_i = S_i^{-1}$ satisfy

$$\big(A(\xi) + BK(\xi)\big)^\intercal P(\xi^+)\big(A(\xi) + BK(\xi)\big) - P(\xi) \preceq -Q - K^\intercal R K \tag{7}$$

for all $\xi, \xi^+ \in \Xi$. This condition bounds the decrease of the Lyapunov function.

In [9], [10] similar LMIs are solved online for robust model predictive control. In model adjustable predictive control [19] the LMI (6) is used to compute the terminal-cost offline. In order to guarantee stability, the decrease in the terminal-cost $x^\intercal P(\xi)x$ for all $\xi \in \Xi$ must be greater than the stage-cost $x^\intercal Qx + u^\intercal Ru$. Thus condition (7) guarantees closed-loop stability of the model predictive controller.

III. Symmetric Linear Parameter Varying Systems

In this section we define symmetry for LPV systems, and parameter-dependent controllers and Lyapunov functions. Then we establish basic results on the existence of symmetric parameter-dependent controllers and Lyapunov functions.

A symmetry of a linear parameter varying system is a state-space $\Theta \in \mathbb{R}^{n \times n}$ and input-space $\Omega \in \mathbb{R}^{m \times m}$ transformation that preserves the dynamics (1).

Definition 1: The pair of invertible matrices (Θ, Ω) is a symmetry of the polytopic linear parameter varying system (1) if

$$\Theta \mathcal{A} = \mathcal{A} \Theta \tag{8a}$$

$$\Theta B = B \Omega. \tag{8b}$$

Condition (8a) implies that for each extreme state-update matrix A_i of $\mathcal{A}$ there exists another extreme state-update matrix $A_j \in \mathcal{A}$ such that $\Theta^{-1} A_i \Theta = A_j$. Definition 1 says that the symmetry (Θ, Ω) maps the extreme dynamics $x(t+1) = A_i x(t) + Bu(t)$ for $i \in \mathcal{I}$ to the extreme dynamics $x(t+1) = \Theta A_i \Theta^{-1} x(t) + \Theta B \Omega^{-1} u(t) = A_j x(t) + Bu(t)$ for $j \in \mathcal{I}$. The set of all symmetries (Θ, Ω) of the LPV system (1) forms an infinite group which we denote by $\mathrm{Aut}(\mathcal{A}, B)$. To see that $\mathrm{Aut}(\mathcal{A}, B)$ is infinite, note that $(\Theta, \Omega) = (\alpha I_n, \alpha I_m)$ is a symmetry for any $\alpha \in \mathbb{R} \setminus \{0\}$. Here we are only interested in symmetries (Θ, Ω) that permute the extreme state-update matrices A_i of the set $\mathcal{A}$ i.e. $\Theta A_i \Theta^{-1} = A_j$ for some $i \neq j \in \mathcal{I}$. These symmetries form a finite subgroup of $\mathrm{Aut}(\mathcal{A}, B)$ which we will denoted by $\mathrm{Perm}(\mathcal{A}, B)$.

Proposition 1: The set $\mathrm{Perm}(\mathcal{A}, B)$ of all symmetries (Θ, Ω) which permute the extreme-points of $\mathcal{A}$ is a finite group.

We will index the elements (Θ_g, Ω_g) of the group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ by $g \in \mathcal{G}$. Since $\mathrm{Perm}(\mathcal{A}, B)$ is finite there exists a basis for the state-space $\mathbb{R}^n$ and input-space $\mathbb{R}^m$ such that Θ_g and Ω_g are orthogonal for all $g \in \mathcal{G}$ [20]. For the remainder of this paper, we will assume Θ_g and Ω_g are orthogonal matrices $\Theta_g^{-1} = \Theta_g^\intercal$ and $\Omega_g^{-1} = \Omega_g^\intercal$.

We define a permutation action $\pi_g : \mathcal{I} \to \mathcal{I}$ for the group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ acting on the index set $\mathcal{I}$ of the extreme state-update matrices $A_i \in \mathcal{A}$ according to the relation $\pi_g(i) = j$ if $\Theta_g A_i \Theta_g^{-1} = A_j$. Thus we can write $\Theta_g A_i \Theta_g^{-1} = A_{\pi_g(i)}$ for all $g \in \mathcal{G}$. The orbits $\mathcal{G}(i) = \{\pi_g(i) \in \mathcal{I} : g \in \mathcal{G}\}$ of the group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ partition the extreme state-update matrices A_i into disjoint equivalence classes. For any extreme state-update matrix A_j with $j \in \mathcal{G}(i)$ there exists a state-space transformation Θ_g for some $g \in \mathcal{G}$ that maps A_i to $A_j = \Theta_g A_i \Theta_g^{-1}$. By transitivity, we can find a state-space transformation Θ_g for some $g \in \mathcal{G}$ that relates any pair of extreme state-update matrices A_j and A_k with $j, k \in \mathcal{G}(i)$.

Example 1 demonstrates the concept of symmetry for linear parameter varying systems.

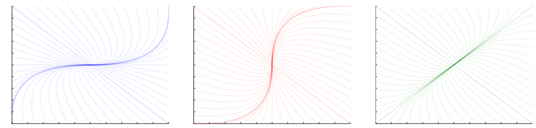
Example 1: Consider a linear parameter varying system of the form (1) with three extreme state-update matrices

$$A_1 = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.8 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.9 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0.8 & 0.1 \\ 0.1 & 0.8 \end{bmatrix}$$

and $B = I \in \mathbb{R}^{2 \times 2}$. Thus $\mathcal{A} = \mathrm{conv}\{A_1, A_2, A_3\}$ and $\mathcal{I} = \{1, 2, 3\}$. The vector-fields $f_i(x) = A_i x$ for the state-update matrices A_i are shown in Figure 1.

This LPV system has one non-trivial symmetry $\Theta_{g_1} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ with $\Omega_{g_1} = \Theta_{g_1}$ satisfying $\Theta_{g_1} B = B \Omega_{g_1}$ since $B = I$. Thus the symmetry group is $\mathcal{G} = \{g_0, g_1\}$ where $\Theta_{g_0} = \Omega_{g_0} = I \in \mathbb{R}^{2 \times 2}$ is the identity matrix.

The state-space transformation Θ_{g_1} maps the state-update matrix A_1 to $A_2 = \Theta_{g_1} A_1 \Theta_{g_1}^{-1}$ and maps A_3 to itself $A_3 = \Theta_{g_1} A_3 \Theta_{g_1}^{-1}$.



(a) $f_1(x) = A_1 x$ (b) $f_2(x) = A_2 x$ (c) $f_3(x) = A_3 x$

Fig. 1. Vector-fields $f_i(x) = A_i x$ of the three extreme state-update matrices A_i for $i \in \mathcal{I} = \{1, 2, 3\}$.

This can be seen in the vector-fields $f_i(x) = A_i x$ shown in Figure 1. The symmetry Θ_{g_1}, which permutes the x and y axis, maps the vector-field $f_1(x) = A_1 x$ to the vector-field $f_2(x) = A_2 x$ and does not change the vector-field $f_3(x) = A_3 x$. Thus $g_1 \in \mathcal{G}$ corresponds to the permutation $\pi_{g_1} = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$. The identity matrix $\Theta_{g_0} = \Omega_{g_0} = I$ corresponds to the identity permutation $\pi_{g_0} = e$.

The index set $\mathcal{I}$ has two orbits $\mathcal{G}(1) = \{1, 2\}$ and $\mathcal{G}(3) = \{3\}$ since the symmetry g_1 relates A_1 and $A_2 = \Theta_{g_1} A_1 \Theta_{g_1}^{-1}$ and A_3 is only related to itself $A_3 = \Theta_{g_0} A_3 \Theta_{g_0}^{-1} = \Theta_{g_1} A_3 \Theta_{g_1}^{-1}$. We can select $\mathcal{I}/\mathcal{G} = \{1, 3\}$ as a set of representative indices from each orbit. The stabilizer subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$, of A_1 and A_2 respectively, are trivial $\mathcal{G}_1 = \mathcal{G}_2 = \{g_0\}$ since only the identity element $\Theta_{g_0} = I$ maps A_1 and A_2 to themselves. The stabilizer group $\mathcal{G}_3$ for A_3 is the entire group $\mathcal{G} = \mathcal{G}_3$ since every symmetry Θ_g for $g \in \mathcal{G} = \{g_0, g_1\}$ maps A_3 to itself $A_3 = \Theta_{g_0}^{-1} A_3 \Theta_{g_0} = \Theta_{g_1}^{-1} A_3 \Theta_{g_1}$. $\square$

Next we define symmetry for parameter-dependent controllers and Lyapunov functions. A parameter-dependent controller (2) is symmetric if the state-space transformation Θ_g and input-space transformation Ω_g preserve $\Theta_g \mathcal{K} = \mathcal{K} \Omega_g$ the controller polytope $\mathcal{K}$ for each $g \in \mathcal{G}$

Definition 2: The parameter-dependent control law (2) is symmetric with respect to the group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ if for every extreme-controller gain $K_i \in \mathcal{K}$ and $g \in \mathcal{G}$

$$\Omega_g K_i \Theta_g^{-1} = K_{\pi_g(i)}. \tag{9}$$

By definition 2 the state-space transformation Θ_g and input-space transformation Ω_g permute the extreme gains K_i of $\mathcal{K}$ in the same way that they permute the extreme state-update matrices A_i. Thus the orbits $\mathcal{G}(i)$ for $i \in \mathcal{I}/\mathcal{G}$ partition the extreme controller gains K_i of a symmetric controller into disjoint equivalence classes. Every pair of extreme controller gains K_j and K_k for $j, k \in \mathcal{G}(i)$ are related $K_j = \Omega_g K_k \Theta_g^{-1}$ by some state-space Θ_g and input-space Ω_g transformation with $g \in \mathcal{G}$.

Similarly we can define symmetry for a parameter-dependent Lyapunov function. Recall that since $\mathrm{Perm}(\mathcal{A}, B)$ is finite, there exists a basis such that Θ_g and Ω_g are orthogonal $\Theta_g^{-1} = \Theta_g^\intercal$ and $\Omega_g^{-1} = \Omega_g^\intercal$.

Definition 3: The parameter-dependent Lyapunov function (3) is symmetric with respect to the group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ if for every extreme-matrix P_i and $g \in \mathcal{G}$

$$\Theta_g P_i \Theta_g^\intercal = P_{\pi_g(i)}. \tag{10}$$

where $\Theta_g^\intercal = \Theta_g^{-1}$ are orthogonal.

By definition 3 the state-space transformation Θ_g permutes the extreme Lyapunov matrices P_i in the same way that they permute the extreme state-update matrices A_i. Thus the orbits $\mathcal{G}(i)$ for $i \in \mathcal{I}/\mathcal{G}$ partition the extreme Lyapunov matrices P_i of a symmetric controller into disjoint equivalence classes. Every pair of extreme matrices P_j and P_k for $j, k \in \mathcal{G}(i)$ are related by a state-space transformation Θ_g for $g \in \mathcal{G}$.

The following theorem shows that symmetric linear parameter varying systems have symmetric parameter-dependent controllers and Lyapunov functions.

Theorem 1: The LPV system (1) has a stabilizing parameter-dependent controller (2) with a parameter-dependent Lyapunov function (3) if and only if it has a symmetric controller and Lyapunov function.

Theorem 1 shows that symmetric LPV systems (1) have symmetric parameter-dependent controllers (2) and Lyapunov functions (3). Next we establish that the LMI (6) permits symmetric solutions. The following lemma shows that the feasible region of the linear matrix inequalities (6) is symmetric.

Lemma 1: Let Q and R satisfy $\Theta_g Q \Theta_g^\intercal = Q$ and $\Omega_g R \Omega_g^\intercal = R$ for all $g \in \mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$. Let E_i, G_i, S_i, and S_j be a solution to the linear matrix inequalities (6) for $(i,j) \in \mathcal{I}^2$. Then $E_{\pi_g(i)} = \Omega_g E_i \Theta_g^{-1}$, $G_{\pi_g(i)} = \Theta_g G_i \Theta_g^{-1}$, $S_{\pi_g(i)} = \Theta_g S_i \Theta_g^{-1}$, and $S_{\pi_g(j)} = \Theta_g S_j \Theta_g^{-1}$ is a solution to (6) for $(\pi_g(i), \pi_g(j)) \in \mathcal{I}^2$.

By Lemma 1 if we have found a solution to the linear matrix inequality (6) for the pair $(i,j) \in \mathcal{I}$, then we have found a solution to (6) for every pair $(\pi_g(i), \pi_g(j))$ in the orbital $\mathcal{G}(i,j) \subseteq \mathcal{I}^2$ of $(i,j) \in \mathcal{I}^2$. Additionally if the set of triples G_i, E_i, and S_i for each $i \in \mathcal{I}$ satisfies the LMI (6) then the set of triples $\Theta_g^{-1} G_{\pi_g(i)} \Theta_g$, $\Omega_g^{-1} E_{\pi_g(i)} \Theta_g$, and $\Theta_g^{-1} S_{\pi_g(i)} \Theta_g$ is also satisifies (6).

The following theorem shows that the linear matrix inequalities (6) permit symmetric parameterizations of the controller gains $K_i = E_i G_i^{-1}$ and Lyapunov matrices $P_i = S_i^{-1}$.

Theorem 2: Suppose Q and R satisfy $\Theta_g Q \Theta_g^\intercal = Q$ and $\Omega R \Omega^\intercal = R$. Then the linear matrix inequalities (6) are feasible if and only if they have a symmetric solution

$$S_{\pi_g(i)} = \Theta_g S_i \Theta_g^{-1}, \; G_{\pi_g(i)} = \Theta_g G_i \Theta_g^{-1}, \quad \text{and } E_{\pi_g(i)} = \Omega_g E_i \Theta_g^{-1} \tag{11}$$

for all $g \in \mathcal{G}$ and $i \in \mathcal{I}$.

Remark 1: Theorem 2 is a generalization of the fixed-space theorem in [21]. If $\mathcal{A} = \{A_i\}$ is a singleton set then Theorem 2 is equivalent to Theorem 3.3 of [21].

Theorem 2 means that we can restrict our search for design variables E_i, G_i, and S_i that satisfy the LMI (6) to symmetric solutions. In the next section we show how restricting our search to symmetric design variables E_i, G_i, and S_i reduces the computational complexity.

IV. Design of Symmetric LPV Controllers

In this section we propose three complementary methods for using symmetry to reduce the complexity of designing controllers for symmetric linear parameter varying systems. The first method reduces the number of design variables. The second method reduces the dimension of the design variables. The third method reduces the number of linear matrix inequalities that the design variables must satisfy.

A. Reduction of the number of design variables

In this section we show how to reduce the number of design variables E_i, G_i, and S_i for $i \in \mathcal{I}$ in the LMI (6).

Using symmetry we only need to define one set of design variables E_i, G_i, and S_i for a representative $i \in \mathcal{I}$ for each orbit $\mathcal{G}(i)$. These design variables must satisfy the linear matrix inequalities

$$\mathrm{LMI}(\Theta_g G_i \Theta_g^{-1}, \Omega_g E_i \Theta_g^{-1}, \Theta_g S_i \Theta_g^{-1}, \Theta_h S_j \Theta_h^{-1}) \succeq 0 \tag{12}$$

for each $i,j \in \mathcal{I}/\mathcal{G}$, $g \in \mathcal{G}/\mathcal{G}_i$, and $h \in \mathcal{G}/\mathcal{G}_j$. After solving the linear matrix inequalities (12) we set

$$E_j = \Omega_g E_i \Theta_g^{-1}, \; G_j = \Theta_g G_i \Theta_g^{-1} \text{ and } S_j = \Theta_g S_i \Theta_g^\intercal \tag{13}$$

for all $j \notin \mathcal{I}/\mathcal{G}$ where $\pi_g(i) = j$ for some $g \in \mathcal{G}$. The parameter-dependent controller (2) for $K_j = E_j G_j^{-1}$ is stabilizing as certified by the Lyapunov function (3) for $P_j = S_j^{-1}$.

The linear matrix inequalities (12) are the original linear matrix inequalities (6) with an additional constraint (13) that the design variables are symmetric (11) for $g \in \mathcal{G}/\mathcal{G}_i$. According to Lemma 1, restricting the design variables to be symmetric does not alter the feasibility of the linear matrix inequalities. The linear matrix inequalities (12) are feasible if and only if the original linear matrix inequalities (6) are feasible.

This design procedure requires solving $|\mathcal{I}|^2$ linear matrix inequalities in $3|\mathcal{I}/\mathcal{G}|$ design variables instead of $3|\mathcal{I}|$ design variables. The reduction in the number of design variables is the result of enforcing the constraint (13) that the design variables are symmetric for $g \in \mathcal{G}/\mathcal{G}_i$. Since E_j, G_j and S_j for $j \in \mathcal{G}(i)$ are then dictated by (13) we can eliminate them from the linear matrix inequalities (12). This method of variable reduction works best when the orbits $\mathcal{G}(i)$ are large $|\mathcal{G}(i)| \approx |\mathcal{G}|$. In this case the number of orbits is small $|\mathcal{I}/\mathcal{G}| \approx |\mathcal{I}|/|\mathcal{G}|$ for a large group $|\mathcal{G}| \gg 1$. Thus the number of design variables E_i, G_i, and S_i is much smaller $3|\mathcal{I}/\mathcal{G}| \ll 3|\mathcal{I}|$. In the next subsection we describe a complexity reduction method that works best when the orbits $\mathcal{G}(i)$ are small $|\mathcal{G}(i)| \ll |\mathcal{G}|$.

The following example demonstrates the reduction in the number of design variables.

Example 2: The LPV system in Example 1 has $|\mathcal{I}| = 3$ extreme state-update matrices. For each extreme state-update matrix A_i we need to design a controller gain $K_i = E_i G_i^{-1}$ and a Lyapunov function matrix $P_i = S_i^{-1}$ for $i = 1, 2, 3$. Thus we have $3|\mathcal{I}| = 9$ design variables .

Since the extreme state-update matrices A_1 and A_2 are related $A_2 = \Theta_{g_1} A_1 \Theta_{g_1}^{-1}$ by the symmetry Θ_{g_1}, we can use the controller gain $K_2 = \Omega_{g_1} K_1 \Theta_{g_1}^{-1}$ and Lyapunov matrices $P_2 = \Theta_{g_1} P_1 \Theta_{g_1}^\intercal$. Thus we can eliminate the design variables E_2, G_2, and S_2. This leaves $3|\mathcal{I}/\mathcal{G}| = 6$ design variables we must find by solving $|\mathcal{I}|^2 = 9$ LMIs (12) where the design variables E_2, G_2, and S_2 have been replaced by $\Omega_{g_1} E_1 \Theta_{g_1}^{-1}$, $\Theta_{g_1} G_1 \Theta_{g_1}^{-1}$, and $\Theta_{g_1} S_1 \Theta_{g_1}^\intercal$ respectively. $\square$

B. Reduction of the dimension of the design variables

In this section we show how to use symmetry to reduce the dimension of the design variables E_i, G_i, and S_i for $i \in \mathcal{I}$. This method of complexity reduction is complementary to the method in Section IV-A since it can be applied to the design variables E_i, G_i, and S_i for $i \in \mathcal{I}/\mathcal{G}$ not eliminated.

The stabilizer subgroup $\mathcal{G}_i$ is the symmetry group of the design variables E_i, G_i, and S_i since $\Theta_g S_i = S_i \Theta_g$, $\Theta_g G_i = G_i \Theta_g$, and $\Omega_g E_i = E_i \Theta_g$ for all $g \in \mathcal{G}_i$ by the definition of $\mathcal{G}_i$. Therefore we can restrict the design variables E_i, G_i, and S_i to the commutator subspaces of the stabilizer subgroup $\mathcal{G}_i$

$$S_i, G_i \in \mathcal{G}_i(\Theta, \Theta) = \{M \in \mathbb{R}^{n \times n} : \Theta_g M = M \Theta_g, \text{ for } g \in \mathcal{G}_i\}$$
$$E_i \in \mathcal{G}_i(\Theta, \Omega) = \{N \in \mathbb{R}^{m \times n} : \Omega_g N = N \Theta_g, \text{ for } g \in \mathcal{G}_i\}.$$

According to Lemma 1, restricting the design variables to be symmetric (11) for $g \in \mathcal{G}_i$ does not alter the feasibility of the original LMIs i.e. (6) has a solution G_i, S_i, and E_i if and only if it has a solution in the commutators $G_i, S_i \in \mathcal{G}_i(\Theta, \Theta)$ and $E_i \in \mathcal{G}_i(\Theta, \Omega)$.

Recall that the commutators $\mathcal{G}_i(\Theta, \Theta) \subseteq \mathbb{R}^{n \times n}$ and $\mathcal{G}_i(\Omega, \Theta) \subseteq \mathbb{R}^{m \times n}$ are vector-spaces of lower dimension than the vectors-spaces $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{m \times n}$ which nominally contain the design variables $S_i, G_i \in \mathbb{R}^{n \times n}$, and $E_i \in \mathbb{R}^{m \times n}$. We can reduce the dimension of the design variables E_i, G_i, and S_i by explicitly expressing them in terms of basis matrices $\mathrm{M}_{ij} \in \mathbb{R}^{n\times n}$ and $\mathrm{N}_{ij} \in \mathbb{R}^{m\times n}$ for $\mathcal{G}_i(\Theta,\Theta)$ and $\mathcal{G}_i(\Omega,\Theta)$ respectively $S_i = \sum_{j=1}^{d_i^n} s_{ij}\mathrm{M}_{ij}$, $G_i = \sum_{j=1}^{d_i^n} g_{ij}\mathrm{M}_{ij}$, and $E_i = \sum_{j=1}^{d_i^m} e_{ij}\mathrm{N}_{ij}$ where d_i^n and d_i^m are the dimensions of the vector-spaces $\mathcal{G}_i(\Theta,\Theta)$ and $\mathcal{G}_i(\Omega,\Theta)$ respectively. The new design variables are the scalar coefficients $s_{ij}, g_{ij}, e_{ij} \in \mathbb{R}$ of the basis-matrices M_{ij} and N_{ij}. Methods exist for constructing the basis-matrices M_{ij} and N_{ij} when Θ_g and Ω_g for $g \in \mathcal{G}_i$ are permutation matrices [17]. For general symmetries Θ_g and Ω_g, the design variables E_i, G_i, and S_i can be block-diagonalized by "symmetry adapting" the basis matrices M_{ij} and N_{ij} [18], [20].

This method works best when the orbits $\mathcal{G}(i)$ are small $|\mathcal{G}(i)| \ll |\mathcal{G}|$. In this case the symmetry group $\mathcal{G}_i$ of the design variables E_i, G_i, and S_i is large $|\mathcal{G}_i| = |\mathcal{G}|/|\mathcal{G}_i|$ when the group is large $|\mathcal{G}| \gg 1$. Thus the commutator spaces $\mathcal{G}_i(\Theta,\Theta) \subseteq \mathbb{R}^{n\times n}$ and $\mathcal{G}_i(\Omega,\Theta) \subseteq \mathbb{R}^{m\times n}$ are low-dimensional. The reduction of the dimensions of the design variables can be made without changing the feasibility of the original linear matrix inequalities (6).

The following example demonstrates the reduction in the dimension of the design variables.

Example 3: Consider the LPV system in Example 1. Since the stabilizer group $\mathcal{G}_1$ of $i = 1 \in \mathcal{I} = \{1,2,3\}$ is trivial $|\mathcal{G}_1| = 1$ we cannot use it to reduce the dimension of the design variables E_1, G_1, and S_1. On the other hand, the non-trivial stabilizer $\mathcal{G}_3 = \mathcal{G}$ of $i = 3 \in \mathcal{I}$ can be used to reduce the dimensions of the design variables E_3, G_3, and S_3. Without changing the feasibility of the linear matrix inequalities (12), we can restrict E_3, G_3, and S_3 to lie in the commutator subspace $\mathcal{G}_i(\Theta,\Theta) = \mathcal{G}_i(\Theta,\Omega)$ of the subgroup $\mathcal{G}_3 = \{g_0, g_1\}$

$$\mathcal{G}_i(\Theta,\Theta) = \left\{ M \in \mathbb{R}^{2\times 2} : \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] M = M \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \right\}$$

where $\mathcal{G}_i(\Theta,\Theta) = \mathcal{G}_i(\Theta,\Omega)$ since $\Theta_g = \Omega_g$ for all $g \in \mathcal{G} = \{g_0, g_1\}$. The commutator subspace $\mathcal{G}_i(\Theta,\Theta) = \mathcal{G}_i(\Theta,\Omega)$ is 2 dimensional with basis matrices

$$\mathrm{M}_{31} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \text{ and } \mathrm{M}_{32} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

Therefore we can write the design variables as

$$E_3 = \left[\begin{smallmatrix} e_{31} & e_{32} \\ e_{32} & e_{31} \end{smallmatrix}\right], \ G_3 = \left[\begin{smallmatrix} g_{31} & g_{32} \\ g_{32} & g_{31} \end{smallmatrix}\right], \ S_3 = \left[\begin{smallmatrix} s_{31} & s_{32} \\ s_{32} & s_{31} \end{smallmatrix}\right].$$

Thus we have reduced the dimension of $E_3 \in \mathbb{R}^{2\times 2}$ and $G_3 \in \mathbb{R}^{2\times 2}$ from 4 to 2 and the dimension of $S_3 = S_3^\intercal \in \mathbb{R}^{2\times 2}$ from 3 to 2. □

C. Reduction of the Number of Linear Matrix Inequalities

In this section we show how to reduce the number of linear matrix inequalities (6) that the design variables E_i, G_i, and S_i must satisfy.

The linear matrix inequalities (6) are related by symmetry. For each $g \in \mathcal{G}$ the linear matrix inequalities for the pair of design variables (i,j) and $(\pi_g(i), \pi_g(j))$ are symmetrically equivalent

$$\mathrm{LMI}(G_{\pi_g(i)}, E_{\pi_g(i)}, S_{\pi_g(i)}, S_{\pi_g(j)}) = \Phi \mathrm{LMI}(G_i, E_i, S_i, S_j)\Phi^\intercal$$

where $\Phi = \mathrm{diag}(\Theta_g, \Theta_g, \Theta_g, \Omega_g)$, and the design variables G_i, E_i, and S_i are symmetric (11). In other words the linear matrix inequality (6) for the pair $(\pi_g(i), \pi_g(j))$ is positive definite if and only if the linear matrix inequality (6) for (i,j) is positive definite. Therefore if the design variables E_i, G_i, and S_i for $i \in \mathcal{I}$ are symmetric (11) then they only need to satisfy the LMI (6) for one pair $(i,j) \in \mathcal{I}^2/\mathcal{G}$ from each orbital $\mathcal{G}(i,j)$. By Lemma 1 this does not change the feasiblity of the original LMIs (6).

We can combine this result with the reduction in the number of design variables presented in Section IV-A. A representative $(i,j) \in \mathcal{I}^2/\mathcal{G}$ from each orbital $\mathcal{G}(i,j)$ can be generated by selecting $i \in \mathcal{I}/\mathcal{G}$ and $j \in \mathcal{I}/\mathcal{G}_i$ where $j = \pi_g(k)$ for some $k \in \mathcal{I}/\mathcal{G}$ and $g \in \mathcal{S}_i$. Thus the reduced design variables E_i, G_i, and S_i for $i \in \mathcal{I}/\mathcal{G}$ must satisfy the linear matrix inequalities

$$\begin{bmatrix} G_i + G_i^\intercal - S_i & (A_iG_i + BE_i)^\intercal & G_i^\intercal & E_i^\intercal \\ (A_iG_i + BE_i) & \Theta_g^\intercal S_k \Theta_g & 0 & 0 \\ G_i & 0 & Q^{-1} & 0 \\ E_i & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0 \tag{15}$$

for each $(i,j) \in \mathcal{I}^2/\mathcal{G}$ where $g \in \mathcal{G}$ and $k \in \mathcal{I}/\mathcal{G}$ such that $j = \pi_g(k)$. The dimension of the design variables E_i, G_i, and S_i for $i \in \mathcal{I}/\mathcal{G}$ can be reduced using the method described in Section IV-B. This design process requires solving $|\mathcal{I}^2/\mathcal{G}|$ linear matrix inequalities (15) in $3|\mathcal{I}/\mathcal{G}|$ design variables .

The following examples demonstrate the reduction in the number of linear matrix inequalities.

Example 4: Consider the LPV system in Example 1. First we need to calculate the orbitals $\mathcal{G}(i,j)$ for all pairs $(i,j) \in \mathcal{I}^2$. There are $|\mathcal{I}^2/\mathcal{G}| = 5$ orbitals $\mathcal{G}(1,1) = \{(1,1),(2,2)\}$, $\mathcal{G}(1,2) = \{(1,2),(2,1)\}$, $\mathcal{G}(1,3) = \{(1,3),(2,3)\}$, $\mathcal{G}(3,1) = \{(3,1),(3,2)\}$, and $\mathcal{G}(3,3) = \{(3,3)\}$. We can select one representative pair (i,j) from each orbital $\mathcal{G}(i,j)$: $\mathcal{I}^2/\mathcal{G} = \big\{(1,1),(1,2),(1,3),(3,1),(3,3)\big\}$. For each pair $(i,j) \in \mathcal{I}^2/\mathcal{G}$ we build a linear matrix inequalities (15) and replace the design variables E_2, G_2, and S_2 with $\Omega_{g_1}^{-1}E_1\Theta_{g_1}$, $\Theta_{g_1}^{-1}E_1\Theta_{g_1}$, and $\Theta_{g_1}^\intercal S_1\Theta_{g_1}$ respectively. Thus we solve $|\mathcal{I}^2/\mathcal{G}| = 5$ linear matrix inequalities in $3|\mathcal{I}/\mathcal{G}| = 6$ design variables instead of the original $|\mathcal{I}^2| = 9$ inequalities in $3|\mathcal{I}| = 9$ design variables . □

V. Application: Building Control

In this section we apply the results of this paper to the control of heating, ventilation, and air conditioning (HVAC) of a building. Symmetry is used to decrease the number of design variables and linear matrix inequalities.

Fig. 2. Building HVAC system with $N = 3$ rooms.

Building HVAC control systems are used to regulate the room temperature in buildings. Figure 2 shows an conceptual diagram of an HVAC system. The HVAC system has a centralized component which includes the compressors, condensers, and ventilation ports and subsystems for each room which include electric heaters, air-conditioner evaporators, ventilation dampers, and the thermal dynamics of the room. The dynamics of an N room building can be model by the following state-space system

$$\underbrace{\begin{bmatrix} x^0(t+1) \\ x^1(t+1) \\ \vdots \\ x^N(t+1) \end{bmatrix}}_{x(t+1)} = \underbrace{\begin{bmatrix} A^{00} & A^{01} & \dots & A^{0N} \\ A^{10} & A^{11} & \dots & A^{1N} \\ \vdots & \vdots & \ddots & \\ A_{N0} & A^{N1} & \dots & A^{NN} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x^0(t) \\ x^1(t) \\ \vdots \\ x^N(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} B^0 & 0 & \dots & 0 \\ 0 & B^1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & B^N \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u^0(t) \\ u^1(t) \\ \vdots \\ u^N(t) \end{bmatrix}}_{u(t)} \tag{16}$$

where $x^0(t) \in \mathbb{R}^{n_0}$ and $x^r(t) \in \mathbb{R}^{n_1}$ for $r = 1,\dots,N$ are the states of the central dynamics and room dynamics respectively and

$u^0(t) \in \mathbb{R}^{m_0}$ and $u^r(t) \in \mathbb{R}^{m_1}$ for $r = 1, \ldots, N$ are the central and room control inputs respectively.

The state-update matrices $A^{rr} = A^{rr}(\rho_r)$ for each room depend on a scalar parameter $\rho^r \in [\underline{\rho}, \overline{\rho}]$ which represents room-size. The coupling matrices $A^{rs} = A^{rs}(\rho_r, \rho_s)$ depend on the relative sizes ρ^r and ρ^s of rooms. The rooms can be different sizes $\rho^r \neq \rho^s$, however we assume that the dynamics are identical $A^{rr}(\rho^r) = A^{ss}(\rho^s)$ if rooms r and s are the same size $\rho^r = \rho^s$ and $A^{rs}(\rho^r, \rho^s) = A^{pq}(\rho^p, \rho^q)$ if $\rho^r = \rho^p$ and $\rho^s = \rho^q$. Additionally we assume the input matrices B^r for each room are identical $B^r = B^s$ for all $r, s = 1, \ldots, N$. We conservatively assume the range of room sizes $[\underline{\rho}, \overline{\rho}]$ for each room is the same $\rho^r \in [\underline{\rho}, \overline{\rho}]$ for $r = 1, \ldots, N$.

The parameter vector $\rho \in \mathbb{R}^N$ collects the room sizes ρ^r for $r = 1, \ldots, N$. The state-update matrix $A(\rho)$ belongs to a polytope $\mathcal{A} = \{A(\rho) \in \mathbb{R}^{n \times n} : \rho \in [\underline{\rho}, \overline{\rho}]^N\}$ where $n = n_0 + N n_1$. This polytope $\mathcal{A}$ has 2^N vertices $A_i = A(\rho_i)$ where each rooms is either small $\rho_i^r = \underline{\rho}$ or large $\rho_i^r = \overline{\rho}$. The actual state-update matrix $A(\rho) = A(\xi) \in \mathcal{A}$ of the system can be written as a convex combination $A(\xi) = \sum_{i \in \mathcal{I}} \xi_i A(\rho_i)$ of the vertex state-update matrices A_i where $\mathcal{I} = \{1, \ldots, 2^N\}$, $\xi \in \Xi \subset \mathbb{R}^{2^N}$, and ρ_i is a vertex of the hypercube $[\underline{\rho}, \overline{\rho}]^N$.

We seek a single parameter-dependent controller $u(t) = K(\xi)x(t)$ to stabilize the temperature of the N rooms for buildings with different rooms sizes $\rho \in [\underline{\rho}, \overline{\rho}]^N$. Using a parameter-dependent controller, the control gain $K(\xi)$ can be adjusted after installation while maintaining closed-loop stability [19]. Using the design procedure from Section II-B, we would need to define design variables E_i, G_i, and S_i for each of the $|\mathcal{I}| = 2^N$ extreme combinations of room size. The design variables would need to satisfy $|\mathcal{I}^2| = 2^{2N}$ linear matrix inequalities (6). This approach is impractical for even moderately sized buildings.

Instead we can use symmetry to achieve an exponential decrease in the number of design variables and number of linear matrix inequalities. The symmetry group $\mathrm{Perm}(\mathcal{A}, B)$ for this problem contains all the matrices of the form

$$\Theta = \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi \otimes I_{n_1} \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} I_{m_0} & 0 \\ 0 & \Pi \otimes I_{m_1} \end{bmatrix} \tag{17}$$

where $\Pi \in \mathbb{R}^{N \times N}$ is any permutation matrix on $\mathbb{R}^N$ and $\otimes$ is the Kronecker product. The group $\mathcal{G} = \mathrm{Perm}(\mathcal{A}, B)$ has $|\mathcal{G}| = N!$ elements. This symmetry group means permuting the rooms does not change the dynamics of all possible combinations of room sizes. We can permute the N rooms without changing the B matrix in (16). The state-update matrix A may change, however the range of possible state-update matrices $\mathcal{A}$ does not change.

The symmetries (17) organize the extreme state-update matrices $A_i = A(\rho_i) \in \mathcal{A}$ into equivalence classes $\mathcal{G}(i)$. Under symmetry, all extreme state-update matrices $A_i = A(\rho_i)$ corresponding to buildings with M small rooms $\rho_i^r = \underline{\rho}$ and $N - M$ large rooms $\rho_i^r = \overline{\rho}$ are equivalent. From each orbit $\mathcal{G}(i)$ we can chose the representative extreme state-update matrix $A_i = A(\rho_i)$ for which the first M rooms are small $\rho_i^r = \underline{\rho}$ for $r = 1, \ldots, M$ and the remaining $N - M$ rooms are large $\rho_i^r = \overline{\rho}$ for $r = M+1, \ldots, N$. Thus we have the following set of representative indices

$$\begin{aligned} \mathcal{I}/\mathcal{G} = \Big\{ & i \in \mathcal{I} : \rho_i^r = \underline{\rho} \text{ for } r = 1, \ldots, M \text{ and } \rho_i^r = \overline{\rho} \text{ for} \\ & r = M+1, \ldots, N \text{ for some } M \in [0, N] \Big\}. \end{aligned} \tag{18}$$

The index set $\mathcal{I}$ has $|\mathcal{I}/\mathcal{G}| = N + 1$ orbits of size $|\mathcal{G}(i)| = \binom{N}{M}$ for $M = 0, \ldots, N$.

According to Section IV-A, we only need to define one set of design variables E_i, G_i, and S_i for one index $i \in \mathcal{I}/\mathcal{G}$ in the representative set $\mathcal{I}/\mathcal{G}$. From the definition (18) of $\mathcal{I}/\mathcal{G}$ this means we can design the controller gains $K_i = E_i G_i^{-1}$ and Lyapunov matrices $P_i = S_i^{-1}$ assuming the rooms are sorted by size. Then for any building containing M small rooms and $N - M$ large rooms we obtain the controller gain K_j and Lyapunov matrix P_j by permuting the room order $K_j = \Omega^{-1} K_i \Theta$ and $P_j = \Theta^\intercal P_i \Theta$ for the controller gain K_i and Lyapunov matrix P_i with the sorted room sizes.

We can also use symmetry to reduce the number of linear matrix inequalities (6). According to Section IV-C we only need to solve the linear matrix inequalities (6) for one representative pair $(i, j) \in \mathcal{I}^2/\mathcal{G}$ from orbital $\mathcal{G}(i, j)$. Using standard techniques from computational group theory we can find pairs $i, k \in \mathcal{I}/\mathcal{G}$ and Θ_g for $g \in \mathcal{G}$ satisfies $\pi_g(k) = j$ for each $(i, j) \in \mathcal{I}^2/\mathcal{G}$. It can be shown that $\mathcal{I}^2/\mathcal{G}$ has size $O(|\mathcal{I}^2/\mathcal{G}|) = N^3$.

Thus using symmetry we have reduced the number of design variables from $O(|\mathcal{I}|) = 2^N$ to $O(|\mathcal{I}/\mathcal{G}|) = N$ and reduced the number of linear matrix inequalities from $O(|\mathcal{I}^2|) = 2^{2N}$ to $O(|\mathcal{I}^2/\mathcal{G}|) = N^3$.

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