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Abstract

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Stability and feasibility of MPC for switched linear systems with dwell-time constraints

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Abstract— **This paper considers the control of discrete-time switched linear systems using model predictive control. A model predictive controller is designed with terminal cost and constraints depending on the terminal mode of the switched linear system. Conditions on the terminal cost and constraints are presented to ensure persistent feasibility and stability of the closed-loop system given sufficiently long dwell-time. A procedure is proposed to numerically compute admissible terminal costs and constraint sets.**

I. Introduction

Model predictive control (MPC) is an effective method for controlling systems with input and state constraints [1], [2]. In MPC, control inputs are obtained by solving a constrained finite-time optimal control problem. This approach allows the explicit consideration of input and state constraints, applies to multi-input multi-output systems, and typically provides good closed-loop performance by optimizing the control input.

Many industrial control problems involve systems with distinct dynamic modes that can be modeled as switched systems. A switched system is a family of dynamic systems with a switching signal specifying which dynamic mode is active as a function of time [3]. Switched linear systems are used in a variety of applications. For instance, in automotive applications driveline dynamics evolve through distinct modes during gearshifts [4]. In heating, ventilation, and air conditioning of buildings, the heating/cooling to a zone may be engaged or disengaged [5], changing the overall system structure. Switched linear models are also commonly used for modeling the dynamics of walking [6], [7].

It is well known that the stability of all individual dynamic modes does not guarantee stability of the overall switched system under arbitrary switching [3]. Different approaches have been proposed in the literature for guaranteeing the stability of such systems. If the frequency of modes switches is limited, this knowledge can often be used to prove stability. The lower-bound on the amount of time that the system spends in each mode is called the dwell-time.

This paper considers the use of model predictive control to stabilize switched systems where input or state constraints must be satisfied. In [8], a model predictive controller was developed for a switched system with an a-priori known switching signal. Stability was established using a variable prediction horizon which planned the input and state trajectories until the next mode transition. This result was extended in [9] where the switching times are unknown but lie in a known interval. In [10], a model predictive controller was proposed for continuous-time nonlinear switched systems with switching signals that were not known a priori. It was shown that if the switched signal satisfied certain dwell-time restrictions and the switching signal could be measured or estimated quickly enough, then the closed-loop system was ultimately bounded.

Analysis and design methods developed for polytopic linear parameter varying systems can be applied to switched linear systems [11], [12]. In fact, switched linear systems can be interpreted as special executions of polytopic linear parameter varying systems, where the dynamics change only between a discrete set of systems in the polytope, including the vertices and possibly some interior points. In [13]–[17] parameter-dependent Lyapunov functions guaranteed stability for system dynamics evolving inside a polytopic set of dynamic systems. However, the use of parameter-dependent Lyapunov functions may be unnecessarily conservative since all dynamics in the polytope are considered, and no restriction is placed on how often the dynamics change.

In this paper, model predictive control is explored for switched linear systems with switching signals known a priori only over a prediction horizon, i.e., there is a finite preview of the switching signal for a fixed number of steps but no information is available afterwards. This applies for instance to gearshifts in vehicles and to zone engagement/disengagement in building control, since in those applications the timing and sequence of the events leading to mode switches is scheduled slightly

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in advance, but no information is available for the "far" future. In the proposed method, the terminal cost and constraints change based on the terminal mode of the switched linear system. Unlike existing work, discrete time systems are considered. The advantage of the proposed MPC lies in a terminal cost and constraints that are selected to ensure persistent feasibility and closed-loop, asymptotic stability. In contrast to existing works, there are no additional requirements, such as a common Lyapunov function or optimization constraints enforcing stability. A procedure is proposed to compute admissible terminal cost functions and constraint sets.

This paper is organized as follows. In Section II switched linear systems, dwell-time, and relevant concepts from constrained control are defined. Section III presentsa switched model predictive controller that selects the optimal control input as a function of the current system state and future evolution of the switching signal over the prediction horizon. The persistent feasibility and asymptotic stability of such model predictive controller are then proved. In Section IV a numerical procedure is proposed for computing a terminal cost and constraints that satisfy the conditions for stability and feasibility. Finally in Section V the control algorithm is demonstrated in a numerical example.

II. Problem Statement

This paper considers the asymptotic stabilization of the following switched discrete-time constrained linear systems,

$$x(t+1) = f_{\sigma(t)}(x(t), u(t)) = A_{\sigma(t)}x(t) + Bu(t). \quad (1)$$

The switching signal, $\sigma : \mathbb{Z} \to \mathcal{M}$, is a known exogenous input to the system that switches the dynamics matrix, $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$, between a finite number of modes, $i \in \mathcal{M} = \{1, \ldots, M\}$. All pairs, (A_i, B) with $i \in \mathcal{M}$, are assumed to be stabilizable.

The state, $x \in \mathbb{R}^n$, and input, $u \in \mathbb{R}^m$, are constrained to compact polytopic sets,

$$x \in \mathcal{X} = \big\{x \in \mathbb{R}^n \mid H_x x \leq F_x\big\} \subset \mathbb{R}^n \quad (2a)$$
$$u \in \mathcal{U} = \big\{u \in \mathbb{R}^m \mid H_u u \leq F_u\big\} \subset \mathbb{R}^m. \quad (2b)$$

The switching signal, σ, is drawn from the set,

$$\Sigma = \big\{\sigma : \mathbb{Z}_{[0,\infty)} \to \mathcal{M} \mid \mathrm{dwell}(\sigma) \geq d\big\},$$

where $t_0 = 0$ and $t_{i+1} = \min\{t > t_i \mid \sigma(t) \neq \sigma(t_i)\}$ are the switching times for the signal σ and the dwell-time, $\mathrm{dwell}(\sigma)$, is the minimum time between switches,

$$\mathrm{dwell}(\sigma) = \min\left\{t_{i+1} - t_i \mid i \in \mathbb{Z}_+\right\}.$$

It is well known that for sufficiently large dwell-times, $\mathrm{dwell}(\sigma) \geq d$, the switched linear system (1) can be stabilized by independently stabilizing each mode, $i \in \mathcal{M}$, using a linear controller, $u(t) = K_i x(t)$ [3]. This paper extends the result to the use of model predictive control by providing conditions that guarantee persistent feasibility and closed-loop stability.

A. Notation and Definitions

For mode $i \in \mathcal{M}$, a set $\mathcal{S}_i \subseteq \mathcal{X}$ is called control invariant if for all $x \in \mathcal{S}_i$ there exist $u \in \mathcal{U}$ such that $A_i x + Bu \in \mathcal{S}_i$. It is positive invariant for the control-law $u = \kappa_i(x)$ if $A_i x + B\kappa_i(x) \in \mathcal{S}_i$ for all $x \in \mathcal{S}_i$.

The set of all states, $x \in \mathcal{X}$, that can be steered into the set $\mathcal{S}$ under the mode $i \in \mathcal{M}$ dynamics (A_i, B) is the pre-set of $\mathcal{S}$,

$$\mathrm{Pre}(\mathcal{S}, i) = \big\{x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } A_i x + Bu \in \mathcal{S}\big\}.$$

The k-step pre-set is defined recursively as

$$\mathrm{Pre}^k(\mathcal{S}, i) = \begin{cases} \mathcal{S} & \text{if } k = 0, \text{ and} \\ \mathrm{Pre}(\mathrm{Pre}^{k-1}(\mathcal{S}, i), i) & \text{if } k \in \mathbb{Z}_{[1,\infty)}. \end{cases}$$

III. Switched Model Predictive Controller

This section presents a switched model predictive controller that uses the current system state, $x(t)$, and future switching sequence, $\sigma(t), \ldots, \sigma(t+N)$, to stabilize the switched linear system (1) while ensuring constraint satisfaction. Persistent feasibility and stability of the closed-loop system are proven for sufficiently long dwell-time.

A. Model Predictive Controller

A switched model predictive controller is designed for the switched constrained linear system (1). The future switching sequence is assumed to be known over the prediction horizon, N, and possibly unknown afterward. The control input, $u(t) = \kappa(x(t), \sigma(t), \ldots, \sigma(t+N))$, to (1) is a function the system state, $x(t)$, and the future switching sequence, $\sigma(t), \ldots, \sigma(t+N)$. The model predictive controller obtains the control input by solving the following constrained finite-time optimal control problem

$$J^\star(x, \sigma_{0|t}, \ldots, \sigma_{N|t}) =$$
$$\min \ x_{N|t}^\intercal P_{\sigma_{N|t}} x_{N|t} + \sum_{k=0}^{N-1} x_{k|t}^\intercal Q x_{k|t} + u_{k|t}^\intercal R u_{k|t} \quad (3a)$$
$$\text{s.t. } x_{k+1|t} = A_{\sigma_{k|t}} x_{k|t} + Bu_{k|t}, \ k \in \mathbb{Z}_{[0,N-1]} \quad (3b)$$
$$x_{k|t} \in \mathcal{X}, \ k \in \mathbb{Z}_{[1,N]} \quad (3c)$$
$$u_{k|t} \in \mathcal{U}, \ k \in \mathbb{Z}_{[0,N-1]} \quad (3d)$$
$$x_{N|t} \in \mathcal{T}_{\sigma_{N|t}, s_t}, \quad (3e)$$

where $x_{0|t} = x(t)$ is the current state of the system (1), $x_{k|t}$ is the predicted state under the control actions $u_{k|t}$ over the horizon N, and $\sigma_{k|t} = \sigma(t+k)$ is the future switching sequence. The cost matrices are positive definite $P_{\sigma_{N|t}}, Q, R \succ 0$. The control input is the first element of the optimal input sequence as a function of the current state,

$$u(t) = u^\star_{0|t}\big(x(t), \sigma(t), \dots, \sigma(t+N)\big). \tag{4}$$

The domain of the controller, (4), is the set $\mathcal{X}_0(\sigma) \subseteq \mathcal{X}$ of initial conditions, $x(t) = x_0 \in \mathcal{X}_0(\sigma)$, for which the optimal control problem (3) has a solution. The set $\mathcal{X}_0(\sigma) = \mathcal{X}_0(\sigma_{0|0}, \dots, \sigma_{N|0}) \subseteq \mathcal{X}$ depends on the initial $N+1$ steps, $\sigma_{0|0}, \dots, \sigma_{N|0}$, of the switching signal, $\sigma \in \Sigma$. The shorthand $J^\star(t) = J^\star(x(t), \sigma_{0|t}, \dots, \sigma_{N|t})$ will be used for the value function at time $t \in \mathbb{Z}_{[0,\infty)}$.

The terminal constraint set, $\mathcal{T}_{\sigma_{N|t}, s_t}$, depends on the terminal mode, $\sigma_{N|t} = \sigma(t+N)$, and the minimum amount of time, s_t, after the predictive horizon N that the system must spend in the terminal mode so that $\sigma \in \Sigma$. This minimum time is calculated as

$$s_t = \min\big\{k \in \mathbb{Z}_{[0,d-1]} \mid \sigma_{N-d+1+j\,|t} = \sigma_{N\,|t}\ \forall j \in \mathbb{Z}_{[k,d]}\big\}.$$

The following assumptions are made about the terminal sets. In Section IV we will provide a method for computing terminal sets $\mathcal{T}_{i,k}$ that satisfy this assumption.

Assumption 1: For each mode, $i \in \mathcal{M}$, the set, $\mathcal{T}_{i,0} \subseteq \mathcal{X}$, is control invariant under the dynamics (A_i, B) and contains the origin in its interior. For $k \in \mathbb{Z}_{[0,d-1]}$ the terminal sets, $\mathcal{T}_{i,k}$, satisfy $\mathcal{T}_{i,k+1} \subseteq \mathrm{Pre}(\mathcal{T}_{i,k}, i)$. For each pair of modes, $i, j \in \mathcal{M}$, the terminal sets satisfy $\mathcal{T}_{i,0} \subseteq \mathcal{T}_{j,d}$.

The terminal sets, $\mathcal{T}_{i,k}$, relax the constraints after a mode switch, $\sigma_{N|t} \neq \sigma_{N|t+1}$. When the terminal mode switches, the condition, $\mathcal{T}_{i,0} \subseteq \mathcal{T}_{j,d}$, ensures that the terminal state, $x_{N|t+1}$, can reach the new terminal set, $\mathcal{T}_{j,d}$, for $x_{N|t} \in \mathcal{T}_{i,0}$. When the system remains in the same terminal mode, $\sigma_{N|t} = \sigma_{N|t+1}$, the terminal constraints become more restrictive over time, $\mathcal{T}_{i,k} \subseteq \mathcal{T}_{i,k+1}$, to ensure that after d time-steps the next mode transition is feasible. In Section III-B we will show that Assumption 1 guarantees the persistent feasibility of the optimal control problem (3).

One potential choice for the terminal sets $\mathcal{T}_{i,k}$ that satisfy Assumption 1 is a common control invariant set $\mathcal{T}_{i,k} = \mathcal{T}$ for $i \in \mathcal{M}$ and $k \in \mathbb{Z}_{[0,d-1]}$. This choice automatically satisfies $\mathcal{T}_{j,k} = \mathcal{T} \subseteq \mathrm{Pre}(\mathcal{T}_{i,k}, i) = \mathrm{Pre}(\mathcal{T}, i)$ since $\mathcal{T}$ is a control invariant set for each set of dynamics. However, this choice can be overly restrictive and may not be possible if the mode dynamics are very different. A less restrictive procedure for choosing the terminal sets will be presented in Section IV

The terminal cost, $x_{N|t}^\intercal P_{\sigma_{N|t}} x_{N|t}$, depends on the terminal mode, $\sigma_{N|t}$. The following assumptions are made about the terminal cost.

Assumption 2: For all $x \in \mathcal{T}_{i,k}$ where $k \in \mathbb{Z}_{[0,d-1]}$, there exists $u \in \mathcal{U}$ such that $f_i(x,u) \in \mathcal{T}_{i,\max\{k-1,0\}}$ and

$$f_i(x,u)^\intercal P_i f_i(x,u) - x^\intercal P_i x + x^\intercal Q x + u^\intercal R u \leq 0. \tag{5}$$

Since $\mathcal{T}_{i,0}$ is control invariant, $\mathcal{T}_{i,0} \subseteq \mathrm{Pre}(\mathcal{T}_{i,0}, i)$. Therefore, Assumption 2 implies that for all $x \in \mathcal{T}_{i,0}$ there exists $u \in \mathcal{U}$ such that (5) holds. This is the standard condition for guaranteeing the stability in model predictive control [1]. Thus Assumption 2 guarantees the stability of each mode $i \in \mathcal{M}$ when mode switching does not occur. In Section III-C we will provide a lower-bound on the dwell-time $\mathrm{dwell}(\sigma)$ that guarantees stability when switching between modes.

B. Persistent Feasibility

In this section the switched linear system (1) in closed-loop with the model predictive controller (4) is shown to be persistently feasible.

Theorem 1: Let Assumption 1 hold. If (3) has a solution for $x(t)$ then it has a solution for $x(t+1) = A_{\sigma(t)}x(t) + Bu(t)$ where $u(t) = u^\star_{0|t}\big(x(t)\big)$.

Proof: Since (3) has a solution at time $t \in \mathbb{Z}_{[0,\infty)}$ we have a feasible input sequence $u^\star_{0|t}, \dots, u^\star_{N-1|t} \in \mathcal{U}$ that generates a state trajectory $x^\star_{1|t}, \dots, x^\star_{N|t}$ that satisfies the dynamics and state constraints

$$x^\star_{k+1|t} = A_{\sigma(t+k)} x^\star_{k|t} + B u^\star_{k|t} \in \mathcal{X}$$

for $k \in \mathbb{Z}_{[0,N-1]}$ where $x_{0|t} = x(t)$ and

$$x^\star_{N|t} \in \mathcal{T}_{\sigma_{N|t}, s_t}.$$

This solution will be used to construct a feasible solution to (3) at time $t+1 \in \mathbb{Z}_{[0,\infty)}$ where $x_{0|t+1} = A_{\sigma(t)}x(t) + Bu^\star_{0|t}$.

First consider the case that $\sigma_{N|t} = \sigma_{N|t+1}$ and $s_t = 0$ i.e. $x_{N|t} \in \mathcal{T}_{\sigma_{N|t},0}$. Since $\mathcal{T}_{\sigma_{N|t},0}$ is control invariant, there exists a feasible input $v \in \mathcal{U}$ such that $A_{\sigma_{N|t}} x_N + Bv \in \mathcal{T}_{\sigma_{N|t},0}$. Thus the input sequence $u_{k-1|t+1} = u^\star_{k|t}$ for $k = \mathbb{Z}_{[1,N-1]}$ and $u_{N-1|t+1} = v$ is a feasible solution to (3).

Next, consider the case that $\sigma_{N|t} = \sigma_{N|t+1}$ and $s_t > 0$ i.e. $x_{N|t} \in \mathcal{T}_{\sigma_{N|t}, s_t} \subseteq \mathrm{Pre}(\mathcal{T}_{\sigma_{N|t}, s_t - 1}, \sigma_{N|t})$. In this case, the index decreases $s_{t+1} = s_t - 1$ from time t to time $t+1$. By definition of $\mathrm{Pre}(\mathcal{T}_{\sigma_{N|t}, s_t-1}, \sigma_{N|t})$, there exists an input, $v \in \mathcal{U}$, such that $A_{\sigma_{N|t}} x_N + Bv \in \mathrm{Pre}(\mathcal{T}_{\sigma_{N|t}, s_t-1}, \sigma_{N|t})$. Thus the input sequence

$u_{k-1|t+1} = u^\star_{k|t}$ for $k = \mathbb{Z}_{[1,N-1]}$ and $u_{N-1|t+1} = v$ is a feasible solution to (3).

Finally, consider the case that $\sigma_{N|t} \neq \sigma_{N|t+1}$, implying $s_t = 0$ and $s_{t+1} = d-1$. Since $\mathcal{T}_{\sigma_{N|t},0} \subseteq \mathcal{T}_{\sigma_{N|t+1},d} \subseteq \mathrm{Pre}(\mathcal{T}_{\sigma_{N|t+1},d-1}, \sigma_{N|t+1})$ there exists an input $v \in \mathcal{U}$ such that $A_{\sigma_{N|t}} x_N + Bv \in \mathcal{T}_{\sigma_{N|t+1},d-1}$. Thus the input sequence $u_{k-1|t+1} = u^\star_{k|t}$ for $k = 1, \ldots, N-1$ and $u_{N-1|t+1} = v$ is a feasible solution to (3). ■

According to Theorem 1, for any initial condition, $x(0) \in \mathcal{X}_0(\sigma)$, in the feasible set $\mathcal{X}_0(\sigma) = \mathcal{X}_0(\sigma_{0|0}, \ldots, \sigma_{N|0})$ of the constrained finite-time optimal control problem (3), the state and input are guaranteed to satisfy the constraints $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ for all switching signals $\sigma(t) \in \Sigma$ and time $t \in \mathbb{Z}_{[0,\infty)}$.

C. Stability

This section demonstrates that for sufficiently long dwell-time, the switched linear system (1) in closed-loop with the model predictive controller (4) is asymptotically stable. First, in Lemma 1, the value function is bounded in terms of the stage cost for any switching signal. This bound is used to prove closed-loop asymptotic stability for switching signals with adequately large dwell times.

Lemma 1: Let $Q \succeq 0$ and let the compact set $\mathcal{D} \subseteq \mathcal{X}_0(\sigma)$ contain the origin in its interior. Then the optimal value function $J^\star(x, \sigma_{0|t}, \ldots, \sigma_{N|t})$ is bounded by the stage-cost

$$x^\intercal Q x \leq J^\star(x, \sigma) \leq \gamma x^\intercal Q x \tag{6}$$

for all $x \in \mathcal{D}$ and all switching signals $\sigma \in \Sigma$ where $1 < \gamma < \infty$.

Proof: See [18]. ■

Next the bound (6) is used to show that, for sufficiently long dwell-time, the the closed-loop system is stable. Unlike traditional model predictive control, the optimal value function $J^\star(x,\sigma)$ of the switched model predictive control problem (3) may increase when the system (1) switches modes. However, the decay of the value function between each switch dominates the jumps in the value function during switching.

Theorem 2: Let the dwell-time d satisfy

$$d > \frac{\log(\gamma)}{\log(1-\gamma^{-1})} + 1. \tag{7}$$

Then the switched linear system (1) in closed-loop with the model predictive controller (4) is asymptotically stable.

Proof: Asymptotic stability will be shown by bounding the optimal value function.

If the terminal mode does not change, $\sigma_{N|t} = \sigma_{N|t+1}$, then using Assumption 2 the value function is bounded as

$$J^\star(t+1) - J^\star(t) \leq -x(t)^\intercal Q x(t). \tag{8}$$

This bound can be simplified using Lemma 1

$$J^\star(t+1) \leq (1-\gamma^{-1}) J^\star(t) \tag{9}$$

for $\sigma_{N|t} = \sigma_{N|t+1}$. If the terminal mode changes $\sigma_{N|t} \neq \sigma_{N|t+1}$ then (8) and Lemma 1 imply that

$$\begin{aligned} J^\star(t+1) &= J^\star(x(t+1), \sigma_{0|t+1}, \cdots, \sigma_{N|t+1}) \\ &\leq \gamma x^\intercal(t+1) Q x(t+1) \\ &\leq \gamma J^\star(x(t+1), \sigma_{0|t+1}, \cdots, \sigma_{N-1|t+1}, \sigma_{N-1|t+1}) \\ &\leq \gamma J^\star(x(t), \sigma_{0|t}, \cdots, \sigma_{N|t}) = \gamma J^\star(t) \end{aligned} \tag{10}$$

for $\sigma_{N|t} \neq \sigma_{N|t+1}$. From (9) and (10), it can be seen that

$$J^\star(t) \leq \gamma^{s(t)} \left(1-\gamma^{-1}\right)^{t-s(t)} J^\star(0),$$

where $s(t)$ is the number of switches up to time t

$$s(t) = \left|\{\tau \in \mathbb{Z}_{[1,t]} \mid \sigma_{N|\tau} \neq \sigma_{N|\tau+1}\}\right|.$$

The number of switches $s(t) < \frac{t}{d} + 1$ is bounded in terms of the dwell-time $d = \mathrm{dwell}(\sigma)$. This implies

$$J^\star(t) < \left(\gamma \left(1-\gamma^{-1}\right)^{d-1}\right)^{\frac{t}{d}} \frac{\gamma^2}{\gamma-1} J^\star(0)$$

where $\gamma(1-\gamma^{-1})^{d-1} < 1$ due to the dwell-time bound (7). Therefore, the value function converges asymptotically to zero since

$$0 \leq \lim_{t\to 0} J^\star(t) \leq \left(\gamma \left(1-\gamma^{-1}\right)^{d-1}\right)^{\frac{t}{d}} \frac{\gamma^2 J^\star(0)}{\gamma-1} = 0.$$

By Lemma 1, this implies that the state converges to the origin since $J^\star(t) \geq x(t)^\intercal Q x(t) \geq 0$ and $Q \succ 0$. Moreover, the closed-loop system is stable; if the initial condition, $x(0)$, satisfies $\|x(0)\|^2 \leq \delta$ where $\delta = \min\left\{r, \epsilon \frac{\underline{\lambda}(Q)}{\overline{\lambda}(Q)} \frac{(\gamma-1)}{\gamma^3}\right\}$ and $r > 0$ is the radius of the stability domain $\mathcal{D} \supseteq \mathcal{B}_r(0)$ then the state trajectory $x(t)$ satisfies $\|x(t)\|^2 \leq \epsilon$ for all $t \in \mathbb{Z}_{[0,\infty)}$ since

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{\underline{\lambda}(Q)} J^\star(t) \leq \frac{1}{\underline{\lambda}(Q)} \frac{\gamma^2}{\gamma-1} J^\star(0) \\ &\leq \frac{1}{\underline{\lambda}(Q)} \frac{\gamma^3}{\gamma-1} x(0)^\intercal Q x(0) \leq \frac{\overline{\lambda}(Q)}{\underline{\lambda}(Q)} \frac{\gamma^3}{\gamma-1} \|x(0)\|^2 \leq \epsilon. \end{aligned}$$

■

IV. Design of Terminal Cost and Constraints

This section presents a numerical procedure for selecting the terminal costs and terminal constraint sets for the constrained optimal control problem (3). We assume the pair (A_i, B) is stabilizable and the state and input constraints are polyhedral. MATLAB toolboxes such as MPT3 [19], YALMIP [20], and SeDuMi [21] can be used for the numerical computations.

The terminal cost matrices P_i are obtained by solving the following linear matrix inequalities

$$\begin{bmatrix} P_i^{-1} & A_iP_i^{-1}+BKP_i^{-1} & 0 & 0 \\ * & P^{-1} & (KP_i^{-1})^{\intercal} & P^{-1} \\ * & * & R^{-1} & \mathbf{0} \\ * & * & * & Q^{-1} \end{bmatrix} \succeq 0, \tag{11a}$$

$$P_i^{-1} \succ 0. \tag{11b}$$

where the decision variables are $P_i^{-1} \in \mathbb{R}^{n\times n}$ and $KP_i^{-1} \in \mathbb{R}^{m\times n}$ for $i \in \mathcal{M}$. The terminal sets $\mathcal{T}_{i,k}$ are chosen using Algorithm 1. Algorithm 1 chooses $\mathcal{T}_{i,0}$ as the maximal positive invariant subset of $\mathcal{D}$ for the linear controller $u = K_ix$

$$\mathcal{T}_{i,0} = \big\{x \in \mathcal{D} \mid (A_i + BK_i)x \in \mathcal{D} \text{ and } K_ix \in \mathcal{U}\big\}$$

for $i \in \mathcal{M}$. The maximal positive invariant subset can be computed using the techniques from [22], [23]. The sets $\mathcal{T}_{i,k}$ for $k \in \mathbb{Z}_{[1,d]}$ are the k-step pre-sets of $\mathcal{T}_{i,0}$ under the controller K_i,

$$\mathcal{T}_{i,k+1} = (A_i + BK_i)^{-1}\mathcal{T}_{i,k} \cap \mathcal{X},$$

where $(A_i + BK_i)^{-1}$ is the pre-image of the matrix $(A_i + BK_i)$. Algorithm 1 reduces the set $\mathcal{D}$ until the sets $\mathcal{T}_{i,k}$ satisfy $\mathcal{T}_{i,0} \subseteq \mathcal{T}_{i,d}$ for $i, j \in \mathcal{M}$. The following proposition shows that the terminal cost and constraints satisfy Assumptions 1 and 2.

Algorithm 1 Computation of terminal sets

```
Set D = X
repeat
    for each mode i ∈ M do
        Set T_{i,0} as the maximal positive invariant set
        for controller K_i in D
        for k = 0, ..., d − 1 do
            Set T_{i,k+1} = (A_i + BK_i)^{-1} T_{i,k} ∩ X as the
            pre-set of T_{i,k}
        end for
    end for
    Set D = ⋂_{i∈M} T_{i,d}
until T_{i,0} ⊆ T_{j,d} for all i, j ∈ M
```

Proposition 1: Suppose Algorithm 1 converges after finitely many iterations. Then the terminal cost P_i and sets $\mathcal{T}_{i,k}$ produced by (11) and Algorithm 1 respectively satisfy Assumptions 1 and 2.

Proof: See [18]. ■

Proposition 1 assumes that Algorithm 1 converges after a finite number of iterations. In future work we will study conditions on the dynamics and constraints which guarantee finite convergence of Algorithm 1.

V. Numerical Example

This section presents a numerical example to illustrate the proposed switched model predictive controller. We consider a switched linear system of the form (1) where

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 1.5 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1.5 \\ 0 & 1.5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix},$$

subject to polytopic constraints of the form (2) where

$$H_x^{\intercal} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix},\ F_x^{\intercal} = \begin{bmatrix} 4 & 4 & 4 & 4 \end{bmatrix},$$
$$H_u^{\intercal} = \begin{bmatrix} 1 & 1 \end{bmatrix},\ F_u^{\intercal} = \begin{bmatrix} 2 & 2 \end{bmatrix}.$$

The stage-cost of the constrained optimal control problem (3) uses the penalty matrices $Q = I$ and $R = 0.164$. The terminal cost matrix P_i and terminal controller K_i were chosen to satisfy the linear matrix inequalities (11) and minimize the cost

$$\operatorname{trace}\big(c_1(P_i^{-1} - I)^{\intercal}(P_i^{-1} - I) + c_2(K_iP_i^{-1})^{\intercal}K_iP_i^{-1}\big)$$

where the constants $c_1, c_2 > 0$ are tuning parameters. The parameter c_1 keeps the terminal cost matrix near identity $P_i \approx I$. The parameter c_2 penalizes the gain of the terminal controller $K_i \approx K_iP_i^{-1}$ for $P_i \approx I$. Minimizing the gain, K_i, increases the region where the terminal controller, $K_ix \in \mathcal{U}$, is valid. For this example, we selected $c_1 = 1$ and $c_2 = 10$. Figures 1 and 2 shown the terminal sets $\mathcal{T}_{i,k}$ generated by Algorithm 1, which converged after 2 iterations.

The switched linear system was controlled using the switched model predictive controller (4) with a prediction horizon of $N = 10$. With our choice of terminal cost, controller, and sets, persistent feasibility and asymptotic stability were guaranteed for some finite dwell-time in a neighborhood of the origin. The domain of the controller $\mathcal{X}_0(\sigma)$ was explored through simulations. For a grid of initial conditions, the closed-loop system was simulated with two different switching signals

$$\sigma_1(t) = \left\lceil \tfrac{t+1}{8} \right\rceil \mod 2 + 1$$
$$\sigma_2(t) = \left\lceil \tfrac{t+9}{8} \right\rceil \mod 2 + 1.$$

Figure 3 shows the initial conditions, $x(0)$, for which the constrained finite-time optimal control problem (3) was feasible. As expected, (3) is feasible for all initial conditions $x(0) \in \bigcap_{i\in\mathcal{M}} \mathcal{T}_{i,0}$ in the intersection of terminal sets $\bigcap_{i\in\mathcal{M}} \mathcal{T}_{i,0}$ regardless of the switching signal $\sigma(t) \in \Sigma$. Outside of this region $\bigcap_{i\in\mathcal{M}} \mathcal{T}_{i,0}$ feasibility of the initial optimal control problem (3) depends on the initial switching signal $\sigma_{0|0}, \ldots, \sigma_{N|0}$. Figure 3 shows the initial states $x(0)$ for which (3) is feasible for the switching signals $\sigma_1(t)$ and $\sigma_2(t)$.

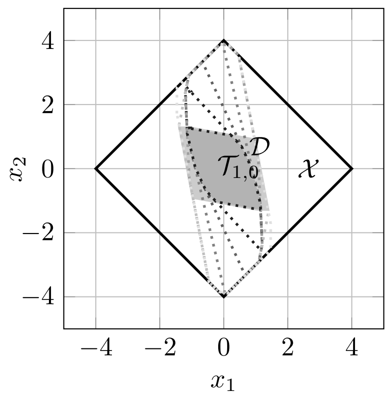

Fig. 1. Terminal sets of mode 1. Dotted regions indicate $\mathcal{T}_{1,k}$ where outlines become lighter as k increases.

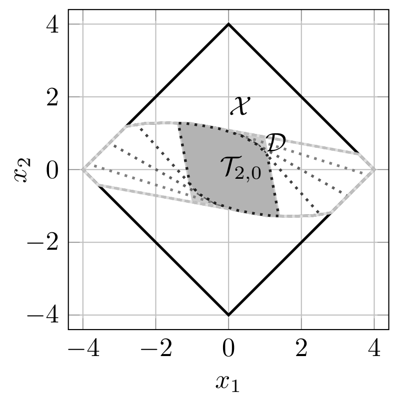

Fig. 2. Terminal sets of mode 2. Dotted regions indicate $\mathcal{T}_{2,k}$ where outlines become lighter as k increases.

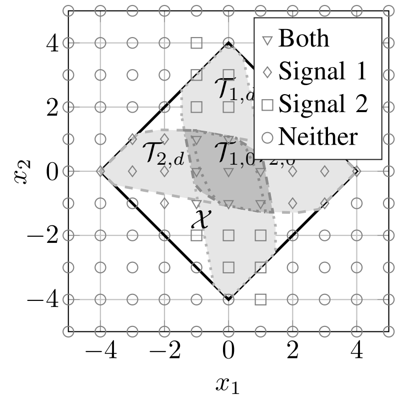

Fig. 3. Markers indicate whether (3) was feasible for various initial conditions with each switching signals tested.

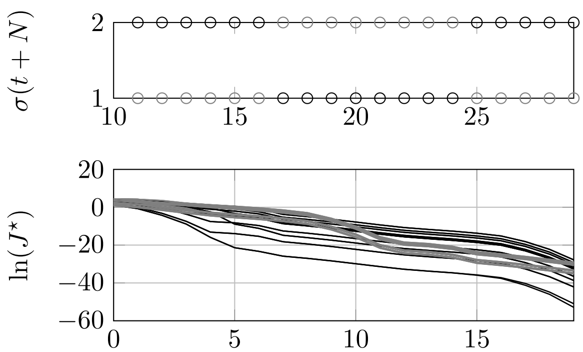

Fig. 4. Trajectories of the mode $\sigma(t)$, value function $J^\star(t)$, and norm $\|x(t)\|_2$ of the state $x(t)$.

Figure 4 shows the cost and the norm of the closed-loop states as functions of time. Semi-log plots were used because convergence was extremely fast. It can be seen that the value function $J^\star(x(t))$ and the closed-loop state $x(t)$ converge to zero as $t \to \infty$.

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