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On extension of a gradient-based co-design algorithm to linear descriptor systems

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***Abstract*—As a holistic approach, optimal co-design of a control system determines both the plant and controller simultaneously to optimize certain performance metrics. Prior co-design work typically assume a control system in the classic state space form, and thus compromise the range of applicability. This paper considers co-design for control systems in the linear descriptor form, with the purpose to minimize the $\mathcal{H}_2$ norm of a closed-loop transfer function. We demonstrate that the gradient of the cost function with respect to plant parameters can be computed analytically, and thus extend the previous gradient-based co-design algorithm to the linear descriptor system case.**

I. Introduction

Control system design needs to determine the physical plant and controller. Compared to sequential design, simultaneous design, or co-design, of the plant and controller may result in improved system performance [1]. The co-design idea has been applied to a wide range of areas, including smart buildings [1], mechatronic systems [2], [3], aerospace crafts [4], and electric motors [5].

Numerous approaches have been proposed to solve co-design problems, for instance co-design of a linear time-invariant (LTI) control system with $\mathcal{H}_2$ or $\mathcal{H}_\infty$ objectives [1], [6], [7] and its nonlinear variant [8], [9], co-design for finite frequency positive-realness property [10], co-design of the controller and underlying communication system [11], etc. See [1], [8] and references therein for details. A majority of existing work assumes that the system model is either in or readily transformable to the classic state space representation

$$\dot{x} = f(x,u), \quad y = h(x,u).$$

This is restrictive for two reasons. First, models of numerous engineering systems, e.g. chemical processes [12], constrained mechanical systems [13], and power systems [14], however naturally take general state space representations such as the descriptor form

$$E\dot{x} = f(x,u), \quad y = h(x,u).$$

Second, for complex systems in the descriptor form, it is not always obvious to derive its classic state space representation. Readers are referred to [15] for more details.

This paper aims to lift this limitation by investigating co-design of control systems in the linear descriptor form. This paper generalizes the gradient-based co-design algorithm originally proposed in previous work [7]. Our main contribution is the derivation of the gradient computation formula for the linear descriptor system, which consequently enables the previous gradient-based co-design algorithm in [7].

The rest of this paper is organized as follows. Section II introduces fundamentals of linear descriptor systems, and formulates the optimal co-design problem. Gradient computation for the linear descriptor system case is derived in Section III. Finally, Section IV offers some future research directions and concludes this paper.

II. Preliminaries and Problem Formulation

In this section, we first introduce fundamentals about linear descriptor systems, and then formulate the co-design problem. Finally, we recite the iterative and alternating co-design procedure [7]. Throughout this paper, dynamical systems are linear time-invariant (LTI) and in continuous-time domain. All matrices are assumed to have compatible dimensions.

A. The Descriptor System and its Solution

Consider an LTI descriptor system

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where $E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_u}, C \in \mathbb{R}^{n_y\times n}$ are constant system matrices, E might be singular and has a rank $r = \text{rank}E \le n$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ the control action, and $y \in \mathbb{R}^{n_y}$ the measurement. The system (II-B) has a unique solution for any initial condition and any continuous input function if $\det(sE - A) \neq 0$ [16], [17]. Equivalently, a system (II-B) with the property $\det(sE - A) \neq 0$ is also called regular. Given the regular pencil $(sE - A)$, its resolvent matrix has the following unique series expansion about $s = \infty$ [18], [19]

$$(sE - A)^{-1} = s^{-1}\sum_{k=-m}^{\infty} \phi_k s^{-k}, \quad m \ge 0, \tag{2}$$

where $\phi_k, -m \le k < \infty$ are the Laurent parameters, and m is the nilpotent index. The series expansion (2) is valid in the set $0 \le |s| \le \delta$ for certain $\delta > 0$. The solution of the descriptor system (II-B) can be expressed directly in terms of the Laurent parameters [18], [19]

$$x(t) = \left(e^{\phi_0 A t x_0} + \int_0^t e^{\phi_0 A(t-\tau)}\phi_0 B u(\tau)\mathrm{d}\tau\right) - \left((-\phi_{-1}E)^m x^{(m)}(t) + \sum_{k=0}^{m-1}(-\phi_{-1}E)^k \phi_{-1} B u^{(k)}(t)\right) \quad (3)$$
$$y(t) = C(\phi_0 E - \phi_{-1}A)x(t).$$

On solutions of discrete-time descriptor systems, see [19], [20] for details.

B. Problem Formulation

Consider a control system in linear descriptor form

$$\begin{aligned} E(\theta)\dot{x} &= A(\theta)x + B_1(\theta)w + B_2(\theta)u \\ z &= C_1(\theta)x + D_{12}(\theta)u \\ y &= C_2(\theta)x + D_{21}(\theta)w, \end{aligned} \quad (4)$$

where $x \in \mathbb{R}^n$ is the state, $\theta \in \mathbb{R}^{n_\theta}$ the plant parameter, $w \in \mathbb{R}^{n_w}$ the external disturbance, $z \in \mathbb{R}^{n_z}$ the regulated output, and $y \in \mathbb{R}^{n_y}$ the measured output. Given inputs $[w^\top, u^\top]^\top$ and outputs $[z^\top, y^\top]^\top$, the system (4) has the following transfer function

$$P(\theta) = \begin{bmatrix} P_{11}(\theta) & P_{12}(\theta) \\ P_{21}(\theta) & P_{22}(\theta) \end{bmatrix}, \quad (5)$$

where $P_{ij} = C_i(sE - A)^{-1}B_j + D_{ij}$. Let T_{wz} be the closed-loop transfer function from w to z. The $\mathcal{H}_2$ optimal control problem is given as follows.

Problem 2.1: Given the system (4) with a fixed θ, find a dynamic output feedback control law $u = Ky$ such that the $\mathcal{H}_2$ norm of T_{wz} is minimized, i.e.,

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & ||T_{wz}(K)||_{\mathcal{H}_2} \\ \text{subject to} \quad & K \text{ stabilizes } P, \end{aligned}$$

where $T_{wz}(K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$.

Provided that E is an identity matrix, the system is in the classic state space form, and the corresponding Problem 2.1 can be solved analytically [21]. For $\mathcal{H}_2$ control of linear descriptor systems (II-B), see [22] and references for details.

Similarly, the optimal plant design induces the optimal plant parameter problem as follows

Problem 2.2: Given the system (4) and its stabilizing controller $u = Ky$, find the plant parameter θ such that the $\mathcal{H}_2$ norm of T_{wz} is minimized, i.e.,

$$\begin{aligned} \underset{\theta}{\text{minimize}} \quad & ||T_{wz}(\theta, K)||_{\mathcal{H}_2} \\ \text{subject to} \quad & \theta \in \Theta, \quad K \text{ stabilizes } P(\theta). \end{aligned}$$

Throughout this paper, we assume that all system matrices in (4) are differentiable with respect to $\theta \in \Theta$ where Θ is compact. We also assume that the pair $[A(\theta), B_2(\theta)]$ is stabilizable and the pair $[A(\theta), C_2(\theta)]$ is detectable for all $\theta \in \Theta$.

The co-design problem is to determine an output feedback control law $u = Ky$ and the plant parameter θ *simultaneously* such that the $\mathcal{H}_2$ norm of the closed-loop transfer function T_{wz} is minimized.

Problem 2.3: Given the system (4), find a dynamic output feedback control law $u = Ky$ and a set of parameter values θ such that the $\mathcal{H}_2$ norm of T_{wz} is minimized, i.e.,

$$\begin{aligned} \underset{\theta, K}{\text{minimize}} \quad & ||T_{wz}(\theta, K)||_{\mathcal{H}_2} \\ \text{subject to} \quad & \theta \in \Theta, \quad K \text{ stabilizes } P(\theta). \end{aligned}$$

Remark 2.4: Problem 2.3 is substantially more difficult to solve compared to Problem 2.1. Actually, when the system matrices in (5) are affine in θ and the controller is state feedback, finding a stabilizing pair (K, θ) is already known to be NP-hard [23]. While in our case, the system matrices are not necessarily affine in θ.

C. Iterative and Alternating Co-design Procedure

This paper follows the same co-design procedure as in [7], i.e., Problem 2.3 is tackled by solving the optimal control problem and the optimal parameter update problem alternately and iteratively. The iterative and alternating co-design procedure is summarized in Algorithm 1, where $\theta^{(k)}$ represents the system parameter in the k-th iteration, $K^{(k)}$ the optimal controller for the system $P(\theta^{(k)})$, and $f(\theta, K)$ abbreviates the objective function $||T_{wz}(\theta, K)||^2_{\mathcal{H}_2}$.

Algorithm 1: Alternating Co-design Algorithm

Choose a threshold $\epsilon \in (0, 1)$;
Initialize $\theta^{(0)} = \theta_0$, $K^{(0)}$ the optimal controller for $P(\theta^{(0)})$;
Set $r = 1$, $k = 0$;
while $r > \epsilon$ **do**
    Calculate $\theta^{(k+1)}$ from $(\theta^{(k)}, K^{(k)})$;
    Compute the optimal controller $K^{(k+1)}$ for $P(\theta^{(k+1)})$;
    Compute $r = \frac{f(\theta^{(k)},K^{(k)}) - f(\theta^{(k+1)},K^{(k+1)})}{f(\theta^{(0)},K^{(0)})}$;
    Set $k = k + 1$;

We assume the initial parameter $\theta^{(0)}$ is given and the initial cost $f(\theta^{(0)}, K^{(0)})$ is finite. Algorithm 1 begins with an initial plant and the corresponding optimal control, and repeats the process of performing the new plant design $\theta^{(k+1)}$ for a given pair $(\theta^{(k)}, K^{(k)})$, and determining the optimal controller $K^{(k+1)}$ for a given $\theta^{(k+1)}$. Particularly, the new plant design is achieved by the plant parameter update algorithm in Appendix A.

Remark 2.5: Convergence analysis of the gradient-based co-design algorithm for the linear descriptor system case can be established as in [7], and thus omitted here. The algorithm convergence is guaranteed by the following fact

1) the cost function is monotonically decreasing in the plant parameter update algorithm
2) the optimal control design for a given $\theta^{(k)}$ also ensures the decrease of the cost function.

Different stopping criteria can be used to ensure the convergence of the co-design algorithm, for instance, the relative improvement ratio $[f(\theta^{(k)}, K^{(k)}) -$

$f(\theta^{(k+1)}, K^{(k+1)})]/f(\theta^{(0)}, K^{(0)})$ is used here. Another stopping criteria is $||\theta^{(k)} - \theta^{(k+1)}||$.

III. Analytical Computation of the Gradient

This section focuses on derivation of an analytical formula of the gradient $\nabla f(\theta)$ used in the plant parameter update algorithm.

Given a state space representation of the $\mathcal{H}_2$ optimal controller

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right], \tag{6}$$

with $A_K \in \mathbb{R}^{n_K \times n_K}, B_K \in \mathbb{R}^{n_K \times n_y}, C_K \in \mathbb{R}^{n_u \times n_K}, D_K \in \mathbb{R}^{n_u \times n_y}$, the closed-loop system is written as follows

$$\begin{aligned} E_c \dot{x}_c &= A_c(\theta, K)x_c + B_c(\theta, K)w \\ z &= C_c(\theta, K)x_c + D_c(\theta, K)w, \end{aligned} \tag{7}$$

where $E_c = \text{diag}\{E, I_{n_K}\}$ with I_{n_K} the n_K-dimensional identity matrix, $x_c = [x^T, x_K^T]^T$, and

$$\begin{aligned} A_c(\theta, K) &= \begin{bmatrix} A(\theta) + B_2(\theta)D_K C_2(\theta) & B_2(\theta)C_K \\ B_K C_2(\theta) & A_K \end{bmatrix} \\ B_c(\theta, K) &= \begin{bmatrix} B_1(\theta) + B_2(\theta)D_K D_{21}(\theta) \\ B_K D_{21}(\theta) \end{bmatrix} \\ C_c(\theta, K) &= \begin{bmatrix} C_1(\theta) + D_{12}(\theta)D_K C_2(\theta) & D_{12}(\theta)C_K \end{bmatrix} \\ D_c(\theta, K) &= D_{12}(\theta)D_K D_{21}(\theta). \end{aligned}$$

The transfer function $T_{wz}(\theta, K)$ has the following concise representation

$$T_{wz}(\theta, K) = \left[\begin{array}{c|c} A_c(\theta, K) & B_c(\theta, K) \\ \hline C_c(\theta, K) & D_c(\theta, K) \end{array}\right] \tag{8}$$

and $T_{wz}(\theta, K) = C_c(sE_c - A_c)^{-1}B_c + D_c$.

As K stabilizes $P(\theta)$, T_{wz} is stable and A_c in (7) is Hurwitz. Denote θ_i the i-th component of θ and $\hat{\theta}_i$ the unit vector in the direction of θ_i. The gradient $\nabla f(\theta)$ can be computed as follows

$$\begin{aligned} \nabla f(\theta) &= \sum_{i=1}^{n_\theta} \frac{\partial < T_{wz}, T_{wz} >}{\partial \theta_i} \hat{\theta}_i \\ &= \sum_{i=1}^{n_\theta} 2 < T_{wz}, \frac{\partial T_{wz}}{\partial \theta_i} > \hat{\theta}_i \end{aligned} \tag{9}$$

where $< .,. >$ is the inner product defined on the $\mathcal{H}_2$ space. Since $T_{wz}(\theta, K) = C_c(sE_c - A_c)^{-1}B_c + D_c$, we have the following formula for $\partial T_{wz}/\partial \theta_i$

$$\begin{aligned} \frac{\partial T_{wz}}{\partial \theta_i} =& \frac{\partial C_c}{\partial \theta_i}(sE_c - A_c)^{-1}B_c + C_c(sE_c - A_c)^{-1}\frac{\partial B_c}{\partial \theta_i} \\ &+ \frac{\partial D_c}{\partial \theta_i} - C_c(sE_c - A_c)^{-1} \\ &\times \left[s\frac{\partial E_c}{\partial \theta_i} - \frac{\partial A_c}{\partial \theta_i}\right](sE_c - A_c)^{-1}B_c \end{aligned} \tag{10}$$

where the last term is derived by considering the following identity [24]

$$\frac{\partial Y^{-1}}{\partial x} = -Y^{-1}\frac{\partial Y}{\partial x}Y^{-1}. \tag{11}$$

The closed-form expression of $\partial T_{wz}/\partial \theta_i$ is derived on the following assumption.

Assumption 3.1: Given E_c and $\frac{\partial E_c}{\partial \theta_i}$ for a fixed θ, there exists a matrix $U_i \in \mathbb{R}^{n+n_K \times n+n_K}$ for $1 \leq i \leq n_\theta$ such that $U_i E_c = \frac{\partial E_c}{\partial \theta_i}$.

Remark 3.2: With $E_c = \text{diag}\{E, I_{n_K}\}$, Assumption 3.1 is equivalent to the existence of a matrix U_{i1} satisfying $U_{i1}E = \partial E/\partial \theta_i$. It is difficult to discuss generally how restrictive Assumption 3.1 is. We have the following conclusions for certain simple cases.

1) If E is diagonal, Assumption 3.1 always holds.
2) If E is linear in θ, i.e., $E(\theta) = \sum_{j=0}^{n_\theta} E_j\theta_j$, Assumption is written as $U_{i1}E(\theta) = E_i$. Without loss of generality, we look at the special case

$$U_{i1}\begin{bmatrix} E_{11}\theta_1 & E_{12}\theta_1 & 0 \\ E_{21}\theta_1 & E_{22}\theta_1 & 0 \\ 0 & 0 & * \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   and conclude that Assumption 3.1 holds and the closed-form solution is $U_{i1} = \text{diag}\{1/\theta_1, 1/\theta_1, 0, \dots\}$. Hence, we know Assumption 3.1 always holds if the system (4) has linear parameterizations.
3) For other cases, Assumption 3.1 may not hold. However, for a loosely coupled system, θ_i appears sparsely in E, and $\partial E/\partial \theta_i$ is sparse. This further implies rank$\partial E/\partial \theta_i \ll$ rank$E$. Hence, Assumption 3.1 is likely satisfied for a fixed value θ. On the other hand, it may be difficult to attain the closed-form solution of U_{i1} and one may need to compute U_{i1} for each different value of θ.

Meanwhile, the matrix U_i might not be unique.

Remark 3.3: Let us consider the following augmented descriptor system

$$\begin{aligned} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} &= \begin{bmatrix} 0 & I_n \\ A(\theta) & -E(\theta) \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ B_1(\theta) \end{bmatrix} w + \begin{bmatrix} 0 \\ B_2(\theta) \end{bmatrix} u \\ z &= C_1(\theta)x + D_{12}(\theta)u \\ y &= C_2(\theta)x + D_{21}(\theta)w. \end{aligned} \tag{12}$$

For the augmented system (12), Assumption 3.1 always holds and can be seemingly lifted. Hence, conclusions in this section are always applicable to (12), and co-design of the system!(4) can be performed on the basis of the augmented system (12). However, compared with the system (4), the augmented system (12) is $2n$ dimension, and its nilpotent index increases by n, which may introduce extra difficulty in solving the co-design problem.

We have the following result about realizations of $\partial T_{wz}/\partial \theta_i$.

Proposition 3.4: Given Assumption 3.1, the transfer function $\partial T_{wz}/\partial\theta_i$ has the following stable realization

$$(I_3 \otimes E_c)\dot{\xi}_i = \begin{bmatrix} A_c & U_iA_c - \frac{\partial A_c}{\partial\theta_i} & 0 \\ 0 & A_c & 0 \\ 0 & 0 & A_c \end{bmatrix} \xi_i + \begin{bmatrix} U_iB_c \\ B_c \\ \frac{\partial B_c}{\partial\theta_i} \end{bmatrix} w$$
$$z_i = \begin{bmatrix} C_c & \frac{\partial C_c}{\partial\theta_i} & C_c \end{bmatrix} \xi_i + \frac{\partial D_c}{\partial\theta_i} w \quad (13)$$

where $I_3 \otimes E_c$ is the Kronecker product, $\xi_i = [\xi_{i3}^T, \xi_{i2}^T, \xi_{i1}^T]^T$, w is defined in (4), and

$$z_i = \frac{\partial T_{zw}}{\partial\theta_i} w.$$

Proof: It is clear that $T_{zw1} = \frac{\partial C_c}{\partial\theta_i}(sE_c - A_c)^{-1}B_c + C_c(sE_c - A_c)^{-1}\frac{\partial B_c}{\partial\theta_i} + \frac{\partial D_c}{\partial\theta_i}$ has a realization

$$\begin{bmatrix} E_c & 0 \\ 0 & E_c \end{bmatrix} \begin{bmatrix} \dot{\xi}_{i2} \\ \dot{\xi}_{i1} \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} \xi_{i2} \\ \xi_{i1} \end{bmatrix} + \begin{bmatrix} B_c \\ \frac{\partial B_c}{\partial\theta_i} \end{bmatrix} w$$
$$z_{i1} = \begin{bmatrix} \frac{\partial C_c}{\partial\theta_i} & C_c \end{bmatrix} \begin{bmatrix} \xi_{i2} \\ \xi_{i1} \end{bmatrix} + \frac{\partial D_c}{\partial\theta_i} w$$

where $z_{i1} = T_{zw1}w$. For a realization of $T_{zw2} = C_c(sE_c - A_c)^{-1}\left[s\frac{\partial E_c}{\partial\theta_i} - \frac{\partial A_c}{\partial\theta_i}\right](sE_c - A_c)^{-1}B_c$, we know that the transfer function T_{zw2} is equivalent to the following dynamics

$$E_c\dot{\xi}_{i3} = A_c\xi_{i3} + \frac{\partial E_c}{\partial\theta_i}\dot{\xi}_{i2} - \frac{\partial A_c}{\partial\theta_i}\xi_{i2}$$
$$E_c\dot{\xi}_{i2} = A_c\xi_{i2} + B_cw$$
$$z_{i2} = C_c\xi_{i3}$$

where $z_{i2} = T_{zw2}w$. Given Assumption 3.1, we have

$$E_c\dot{\xi}_{i3} = A_c\xi_{i3} + U_iE_c\dot{\xi}_{i2} - \frac{\partial A_c}{\partial\theta_i}\xi_{i2}$$
$$= A_c\xi_{i3} + \left(U_iA_c - \frac{\partial A_c}{\partial\theta_i}\right)\xi_{i2} + U_iB_cw$$

With the definition of ξ_i, we have (13). Also, the stability of the system (13) is implied by that of $(sE_c - A_c)^{-1}$. Proposition is therefore shown. ■

Remark 3.5: If the system (4) is in state space form, we can compute $\partial T_{wz}/\partial\theta_i$ as follows

$$\frac{\partial T_{wz}}{\partial\theta_i} = \frac{\partial C_c}{\partial\theta_i}(sI - A_c)^{-1}B_c + C_c(sI - A_c)^{-1}\frac{\partial B_c}{\partial\theta_i} + \frac{\partial D_c}{\partial\theta_i} + C_c(sI - A)^{-1}\frac{\partial A_c}{\partial\theta_i}(sI - A_c)^{-1}B_c. \quad (14)$$

The state space representation of (14) is given by (13) with $E_c = I$, $U_i = 0$, which is consistent with results established in [7].

Given the realization

$$\partial T_{wz}/\partial\theta_i = \left[\begin{array}{c|c} \mathcal{A}_i & \mathcal{B}_i \\ \hline \mathcal{C}_i & \mathcal{D}_i \end{array}\right]$$

we are ready to compute the inner product in (9). For an LTI system in state space representation, with $D_c = 0$, the inner product can be computed in the state space form by forming a Lyapunov-like equation, similar to the computation of the $\mathcal{H}_2$ norm in [25]. For the descriptor system case, we have the following conclusion about the inner product.

Proposition 3.6: Assume the resolvent matrices of the regular pencils $sE_c - A_c$ and $s\mathcal{E}_i - \mathcal{A}_i$ have the unique series expansion given by

$$(sE_c - A_c)^{-1} = s^{-1}\sum_{k=-m}^{\infty}\phi_k s^{-k}, \quad m \geq 0$$
$$(s\mathcal{E}_i - \mathcal{A}_i)^{-1} = s^{-1}\sum_{k=-m_i}^{\infty}\phi_{k,i} s^{-k}, \quad m_i \geq 0.$$

The inner product $< T_{wz}, \frac{\partial T_{wz}}{\partial\theta_i} >$ can be computed from solving

$$\phi_0A_cL + L\mathcal{A}_i^T\phi_{0,i}^T = -\eta(0) \quad (15a)$$
$$< T_{wz}, \frac{\partial T_{wz}}{\partial\theta_i} > = \text{trace}\ \left(C(\phi_0E_c - \phi_{-1}A_c) \times L(\phi_{0,i}\mathcal{E}_i - \phi_{-1,i}\mathcal{A}_i)^TC_i^T\right) \quad (15b)$$

where $\eta(0) = \phi_0B_c\mathcal{B}_i^T\phi_{0,i}^T + \phi_{-1}B_c\mathcal{B}_i^T\phi_{-1,i}^T - \phi_{-1}B_c\mathcal{B}_i^T\phi_{0,i}^T - \phi_0B_c\mathcal{B}_i^T\phi_{-1,i}^T$.

Proof: From the unique series expansion of the resolvent matrices $(sE_c - A_c)^{-1}$ and $(s\mathcal{E}_i - \mathcal{A}_i)^{-1}$, we have the impulse responses of the closed-loop system (7) and the system (13) as follows

$$x_c(t) = e^{\phi_0A_ct}\phi_0B_c - \phi_{-1}B_c$$
$$\xi_i(t) = e^{\phi_{0,i}\mathcal{A}_it}\phi_{0,i}\mathcal{B}_i - \phi_{-1,i}\mathcal{B}_i.$$

The inner product is alternatively defined by $trace(\int_0^\infty y_c(t)y_i^T(t)\mathrm{d}t)$, where y_c and y_i are the output of the closed-loop system (7) and the system (13), respectively, and are given by

$$y_c(t) = C_c(\phi_0E_c - \phi_{-1}A_c)x_c(t)$$
$$y_i(t) = C_i(\phi_{0,i}\mathcal{E}_i - \phi_{-1,i}\mathcal{A}_i)\xi_i(t).$$

With $L = \int_0^\infty x_c(t)\xi_i^T(t)\mathrm{d}t$, we have the inner product given by (15b). We introduce a matrix $\eta(t) = x_c(t)\xi_i^T(t)$ and differentiate it

$$\dot{\eta}(t) = \phi_0A_c\eta(t) + \eta(t)\mathcal{A}_i^T\phi_{0,i}^T$$

Integrating the above equation over $[0, \infty)$ gives

$$\eta(\infty) - \eta(0) = \phi_0A_cL + L\mathcal{A}_i^T\phi_{0,i}^T.$$

Since both the closed-loop system (7) and the auxiliary system (13) are stable, $x(\infty)$, $\xi_i(\infty)$, and $\eta(\infty)$ are zero. We therefore have (15a). This completes the proof. ■

Remark 3.7: For the state variable case, the inner product can be computed by

$$< T_{wz}, \frac{\partial T_{wz}}{\partial\theta_i} >= \text{trace}(C_cLC_i^\top)$$

where L is the solution of the equation

$$A_cL + L\mathcal{A}_i^\top + B_c\mathcal{B}_i^\top = 0. \quad (16)$$

When the plant and controller are discrete time systems, instead of (16), we solve a discrete time Lyapunov-like equation given by

$$A_c L \mathcal{A}_i^\top - L + B_c \mathcal{B}_i^\top = 0.$$

Remark 3.8: Both (15) and (16) can be solved algebraically by the technique of vectorization.

Remark 3.9: The aforementioned gradient computation neglects constraints on θ, and thus the resultant gradient-based plant parameter update algorithm typically requires projection (see Appendix A for details). Alternative to deal with constraints is the augmented Lagrangian approach, which first constructs an augmented Lagrangian as follows

$$\mathcal{L}(\theta, K, \lambda) = f(\theta, K) + \lambda h(\theta)$$

with λ the Lagrangian multiplier and $h(\theta)$ the constraints on θ. Both λ and $h(\theta)$ are vectors with compatible dimensions. In addition to θ, the plant parameter update algorithm also needs to update λ. For a gradient-based update algorithm, the gradient of $\mathcal{L}(\theta, K, \lambda)$ w.r.t. θ and λ is required, which can be similarly computed.

IV. Conclusion and Future Work

This paper investigated co-design of control systems in the linear descriptor form. We shown that the gradient of the cost function w.r.t. plant parameter can be computed analytically, and thus the previous gradient-based co-design algorithm can be readily extended to the linear descriptor system case. Future work includes the application of results to engineering examples and the development of co-design methods for systems in a more general form, for instance nonlinear descriptor systems.

Appendix

A. Plant Parameter Update Algorithm

Given a fixed controller K, our goal is to find a θ_{new} such that the closed-loop system performance has a *sufficient improvement*. The plant parameter update is given by the rule

$$\theta_{new} = \mathbb{P}_\Theta(\theta + \alpha p) \tag{17}$$

where $\alpha \in \mathbb{R}^+$ represents the step length, p the search direction, and $\mathbb{P}_\Theta(.)$ the projection operator on the set Θ. We use the steepest descent direction $p = -\nabla f(\theta)$.

Given the gradient of the cost function w.r.t. plant parameter, a number of algorithms are available to determine α. See [26] for details. A common way to choose the step length α is by the Wolfe conditions [26] given by

$$f(\theta_{new}) \leq f(\theta) + c_1 \alpha \nabla f(\theta)^\top p \tag{18}$$

$$\nabla f(\theta_{new})^\top p \geq c_2 \nabla f(\theta)^\top p \tag{19}$$

for some $0 < c_1 < c_2 < 1$. Equation (18) ensures that the search gives an improvement when p is a descent direction. Note that (18) will be satisfied for any small α, in which case the improvement may be limited. Therefore, the curvature condition (19) is added to ensure that the chosen step length is not too conservative. In practice, we can use the backtracking technique to dispense the condition (19) [26]. The idea is to set α to a large value initially, and decrease its value until (18) is satisfied. When $\Theta = \{\theta | \theta_{min} \leq \theta \leq \theta_{max}\}$ and $p \neq 0$, there exists a finite $\bar{\alpha}$ such that $\mathbb{P}_\Theta(\theta + \alpha p) = \mathbb{P}_\Theta(\theta + \bar{\alpha} p)$ for all $\alpha \geq \bar{\alpha}$. The backtracking technique can start with this initial value. The complete plant parameter update algorithm with backtracking is given as Algorithm 2.

Algorithm 2: Plant Parameter Update Algorithm

```
Given θ, K, f(θ) = ||T_wz(θ, K)||²_H2 ;
Compute the gradient ∇f(θ) and set p = −∇f(θ) ;
if p = 0 then
    θ_new = θ ;
else
    Compute ᾱ and set α = ᾱ ;
    Choose ρ ∈ (0, 1), c1 ∈ (0, 1) ;
    Set θ_new = P_Θ(θ + αp) ;
    while f(θ_new) > f(θ) + c1 α ∇f(θ)^⊤ p do
        α = ρα ;
        θ_new = P_Θ(θ + αp) ;
```

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