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TR2017-039 March 30, 2017

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IEEE Signal Processing Letters

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***Index Terms*— Parameter estimation, polynomial phase signal, frequency modulation, Cramér-Rao bounds.**

I. Introduction

Parameter estimation of pure polynomial phase signals (PPSs) from a finite number of samples is a fundamental problem in many applications, including radar, sonar, communications, acoustics and optics [1]–[20]. A generalized signal model is the independent mixture of PPS and sinusoidal frequency modulated (FM) signal, also referred to as the hybrid sinusoidal FM-PPS in the literature [21]–[27]. One motivation to study this mixture signal comes from Doppler radar systems. When a target is moving in a dynamic motion, outputs at the matched filter can be modeled as a pure PPS with the phase parameter associated to the kinematic parameters of the moving target. For instance, the initial velocity and acceleration are proportional to the first-order and second-order polynomial phase parameters, respectively. On the other hand, rotating parts (e.g., rotating blades of a helicopter) and target vibration introduce the sinusoidal FM component [21]–[24]. With both effects, the matched filter outputs follow the independent mixture signal model.

In this paper, motivated by real-world applications, e.g., contactless linear encoders with periodic reflectors, we propose a new *coupled* mixture of the PPS and sinusoidal FM signal which further generalizes the two models mentioned above. Specifically, the coupling is introduced to express the sinusoidal FM frequency as a function of the PPS parameters. Given the coupled mixture model, we derive the Cramér-Rao bound (CRB) for parameter estimation. On one hand, the derived CRB is dependent on both PPS-related and sinusoidal FM-related parameters, different from the pure PPS case where the CRB is independent of the PPS-related parameters. On the other hand, the derived CRB for the PPS-related parameters are lower than their counterparts of the independent mixture model, as the sinusoidal FM frequency provides additional information on the PPS-related parameters. Numerical examples are also provided to compare the CRBs for different signal models and to show the achievability of the derived CRB.

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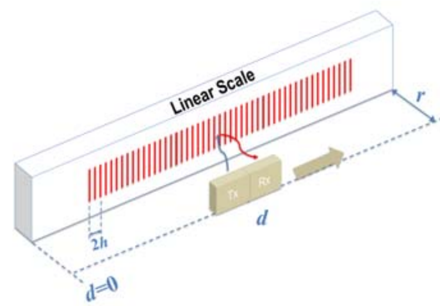

Fig. 1. The geometric configuration of linear encoder systems. The source transceivers are mounted on a moving readhead with a distance to the scale platform. On the scale platform, uniformly spaced bars are installed.

II. A Coupled Mixture Signal Model

This section reviews a specific application to motivate the coupling between the PPS and the sinusoidal FM signal, introduces the coupled mixture model, and compares it with the two existing models.

A. Linear Optical, Electric and Magnetic Encoders

An encoder is an electromechanical device that can monitor motion or position. Among others, optical, electric and magnetic encoders are commonly used for high accuracy motion and position measurements [28], [29]. It normally consists of a stationary scale and a moving readhead, or vice versa; see Fig. 1. The source transceivers are mounted on the moving readhead with a distance of r to the scale platform. Uniformly spaced reflectors, e.g., rectangular bars, are installed on the scale platform to constitute a spatial period with an inter-reflector spacing of h. Then the position and speed of the moving readhead can be inferred from reflected signals in different approaches. One of those approaches is a phase detection approach which detects the phase from the reflected signal. Generally, the reflected signals from the spatially periodic linear scale can be written as

$$x(d) = Ae^{j2\pi\left[\frac{d}{h}+\sum_{m=1}^{M} b_m \sin\left(\frac{2\pi m d}{h}+\phi_m\right)+\psi_0\right]}, \tag{1}$$

where A is the unknown amplitude, d is the axial position index of the moving readhead, $b_m > 0$ and ϕ_m are the modulation index and, respectively, the initial phase of the m-th sinusoidal FM component, M is the number of sinusoidal FM components in the phase, and ψ_0 is the initial phase. The first phase term is due to the phase change proportional to the inter-reflector spacing of h. Therefore, the moving distance and speed of the moving readhead can be inferred from the change in the first phase term. Meanwhile, the second term is, induced by the spatially periodic reflectors, the motion-related sinusoidal FM component. From (1), we have $x(d) = x(d + lh)$, where l is an integer. That is the moving readhead *sees* exactly the same reflected waveforms at two axial positions which are at a distance of h apart from each other.

With a sampling interval of ΔT and assuming that the readhead moves at an initial velocity of v_0 and an acceleration of a, we can transform the position index to the discrete-time index via $d = v_0 t + at^2/2|_{t=n\Delta T} = v_0 n\Delta T + a(n\Delta T)^2/2, n = n_0, \cdots, n_0 + N - 1$ with n_0 and N denoting the initial sampling index and the number of total samples, respectively. As a result, the discrete-time reflected signal is given as

$$\begin{aligned} x(n) =& Ae^{j2\pi\left[\frac{v_0 n\Delta T + a(n\Delta T)^2/2}{h}+\psi_0\right]} \\ &\times e^{j\sum_{m=1}^{M} 2\pi\left[b_m \sin\left(2\pi m \frac{v_0 n\Delta T + a(n\Delta T)^2/2}{h}+\phi_m\right)\right]}. \end{aligned} \tag{2}$$

Note that the sinusoidal FM frequency is now a function of the motion-related phase parameter (e.g., v_0 and a) of the moving readhead.

B. The Coupled Mixture of PPS and Sinusoidal FM Signal

For more dynamic motions of the readhead, higher-order phase terms may appear in the reflected signal. For instance, if the acceleration is time-varying, a third-order phase term (on t^3) may be required to model the reflected signal, i.e., $d = v_0 t + at^2/2 + gt^3/6$ where g denotes the acceleration rate. To generalize the coupled signal model, we propose here a *coupled* mixture of the PPS and sinusoidal FM signals:

$$x(n) = Ae^{j2\pi\left[\sum_{p=0}^{P}\frac{a_p n^p}{p!}+\sum_{m=1}^{M} b_m \sin(2\pi m f_0(a_1,\cdots,a_P)n+\phi_m)\right]}, \tag{3}$$

where the fundamental sinusoidal FM frequency f_0 is now coupled with the PPS phase parameters, $a_1, \cdots, a_P$. Depending on applications, the coupling function $f_0(a_1, \cdots, a_P)$ can be either nonlinear or linear with respect to $\{a_p\}_{p=1}^P$. In the case of linear encoders, it is a linear function as $f_0(a_1, \cdots, a_P) = c_0 \sum_{p=1}^P a_p n^{p-1}/p!$ with c_0 denoting a *known* scaling factor.

To see how the linear encoder example fits into the coupled mixture, we can establish the following variable changes between (2) and (3)

$$b_m = b_m, \quad a_0 = \psi_0, \quad a_1 = \frac{v_0 \Delta T}{h}, \quad a_2 = \frac{a(\Delta T)^2}{h}, \tag{4}$$

$$f_0(a_1, a_2) = \frac{v_0 \Delta T}{h} + \frac{a(\Delta T)^2}{h} n/2 = c_0(a_1 + a_2 n/2),$$

with $c_0 = 1$ and a PPS order of $P = 2$.

To the best of our knowledge, such a coupled mixture model of (3) has never been considered before in the literature. It is distinct from the independent mixture model [21]–[27]

$$x(n) = Ae^{j2\pi\left[\sum_{p=0}^{P}\frac{a_p n^p}{p!}+\sum_{m=1}^{M} b_m \sin(2\pi m f_0 n+\phi_m)\right]}, \tag{5}$$

where the FM frequency f_0 is *independent* of the PPS parameters $\{a_p\}_{p=1}^P$. Second, it generalizes the pure PPS model $x(n) = Ae^{j2\pi \sum_{p=0}^{P} \frac{a_p}{p!} n^p}$ as a special case when $b_m = 0$.

III. Cramér-Rao Bounds for The Coupled Signal

The problem of interest is to estimate the motion-related parameters $\{a_p\}_{p=1}^P$ from a finite number of noisy samples $y(n) = x(n) + v(n)$, where $x(n)$ is given in (3) and $v(n)$ is assumed to be Gaussian distributed with zero mean and variance σ^2. In certain applications, other parameters, e.g., A, $\{b_m\}_{m=1}^M$ and ϕ_m, may be of interest as well. In the following, we derive the performance bound in terms of the CRBs for any unbiased estimator of these unknown parameters.

By defining the following vectors $\mathbf{y} = [y_{n_0}, y_{n_0+1}, \cdots, y_{n_0+N-1}]^T = \mathbf{x}(\boldsymbol{\theta}) + \mathbf{v}$ where $\mathbf{x}$ and $\mathbf{v}$ are similarly defined as $\mathbf{y}$, and $\mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I}_N)$, and $\boldsymbol{\theta} = [Na_1, N^2a_2, \cdots, N^P a_P, a_0, b_1, \cdots, b_M, \phi_1, \cdots, \phi_M]^T$ groups $(P+2M+1)$ unknown real-valued phase parameters[1]. For an unbiased estimator of $\boldsymbol{\theta}$, the CRB provides a lower bound on the variance: $\mathrm{cov}(\hat{\boldsymbol{\theta}}) \geq \mathbf{F}^{-1}$, where $\mathbf{F}$ is the Fisher information matrix (FIM) whose entries involve the derivatives of the log-likelihood function of the samples.

A. Exact CRBs

It is easy to see that $\mathbf{y} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathcal{CN}\left(\mathbf{x}(\boldsymbol{\theta}), \sigma^2\mathbf{I}_N\right)$ where $\boldsymbol{\mu}$ denotes the mean and $\boldsymbol{\Sigma}$ denotes the covariance matrix. Note that $\boldsymbol{\Sigma}$ here is not a function of the parameters $\boldsymbol{\theta}$. To compute the FIM, we use the Slepian-Bangs formula [30] for the following general expression

$$\begin{aligned} \mathbf{F}_{i,j} &= E\left\{\frac{\partial \log p(\mathbf{y}|\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial \log p(\mathbf{y}|\boldsymbol{\theta})}{\partial\theta_j}\right\} \\ &= 2\Re\left[\frac{\partial\boldsymbol{\mu}^H}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_j}\right] + \mathrm{Tr}\left\{\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_i}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_j}\right\}. \end{aligned} \tag{6}$$

Since $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_N$ and $\boldsymbol{\mu} = \mathbf{x}(\boldsymbol{\theta})$, the FIM is simplified to

$$\mathbf{F}_{i,j} = \frac{2}{\sigma^2}\Re\left[\frac{\partial\mathbf{x}^H(\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial\mathbf{x}(\boldsymbol{\theta})}{\partial\theta_j}\right]. \tag{7}$$

Defining $\eta_m = 2\pi c_0 b_m$ and with the following results,

$$\frac{\partial x_n(\boldsymbol{\theta})}{\partial N^p a_p} = Ae^{j\phi_c(n)}\left[1 + \sum_{m=1}^{M}\eta_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right] j\frac{2\pi n^p}{p! N^p},$$

[1]To better expose the derivation of CRBs, we assume the signal amplitude A and the noise variance σ^2 are known. Later we will derive the joint CRBs when the two parameters are unknown.

$$\frac{\partial x_n(\boldsymbol{\theta})}{\partial a_0} = Ae^{j\phi_c(n)}\left[j2\pi\right],$$
$$\frac{\partial x_n(\boldsymbol{\theta})}{\partial b_m} = Ae^{j\phi_c(n)}\left[j2\pi\sin\left(\phi_m^{\mathrm{FM}}(n)\right)\right],$$
$$\frac{\partial x_n(\boldsymbol{\theta})}{\partial \phi_m} = Ae^{j\phi_c(n)}\left[j2\pi b_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right],$$

where $\phi_c(n)$ is the phase of $x(n)$ and $\phi_m^{\mathrm{FM}}(n) = 2\pi m c_0 \sum_{p=1}^{P}\frac{a_p n^p}{p!} + \phi_m$ denotes the phase function of the m-th sinusoidal FM component, we have

$$F_{a_p,a_q} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{N^p\partial a_p}\frac{\partial \mathbf{x}}{\partial N^q a_q}\right], p,q=1,2,\cdots,P, \quad (8)$$
$$= \frac{8\pi^2|A|^2}{p!q!\sigma^2}\sum_{n\in\mathbb{N}}\left[1+\sum_{m=1}^{M}\eta_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right]^2\left(\frac{n}{N}\right)^{p+q},$$

$$F_{a_0,a_0} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial a_0}\frac{\partial \mathbf{x}}{\partial a_0}\right] = \frac{8\pi^2|A|^2}{\sigma^2}N, \quad (9)$$

$$F_{b_m,b_{m'}} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial b_m}\frac{\partial \mathbf{x}}{\partial b_{m'}}\right], \quad (10)$$
$$= \frac{8\pi^2|A|^2}{\sigma^2}\sum_{n\in\mathbb{N}}\sin\left(\phi_m^{\mathrm{FM}}(n)\right)\sin\left(\phi_{m'}^{\mathrm{FM}}(n)\right),$$

$$F_{\phi_m,\phi_{m'}} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial \phi_m}\frac{\partial \mathbf{x}}{\partial \phi_{m'}}\right], \quad (11)$$
$$= \frac{8\pi^2|A|^2 b_m b_{m'}}{\sigma^2}\sum_{n\in\mathbb{N}}\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\cos\left(\phi_{m'}^{\mathrm{FM}}(n)\right),$$

$$F_{a_p,a_0} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial N^p a_p}\frac{\partial \mathbf{x}}{\partial a_0}\right], \quad (12)$$
$$= \frac{8\pi^2|A|^2}{p!\sigma^2}\sum_{n\in\mathbb{N}}\left[1+\sum_{m=1}^{M}\eta_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right]\left(\frac{n}{N}\right)^{p},$$

$$F_{a_p,b_m} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial N^p a_p}\frac{\partial \mathbf{x}}{\partial b_m}\right] = \frac{8\pi^2|A|^2}{p!\sigma^2}, \quad (13)$$
$$\times\sum_{n\in\mathbb{N}}\left[1+\sum_{m=1}^{M}\eta_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right]\sin\left(\phi_m^{\mathrm{FM}}(n)\right)\left(\frac{n}{N}\right)^{p},$$

$$F_{a_p,\phi_m} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial N^p a_p}\frac{\partial \mathbf{x}}{\partial \phi_m}\right] = \frac{8\pi^2|A|^2 b_m}{p!\sigma^2}, \quad (14)$$
$$\times\sum_{n\in\mathbb{N}}\left[1+\sum_{m=1}^{M}\eta_m\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\right]\cos\left(\phi_m^{\mathrm{FM}}(n)\right)\left(\frac{n}{N}\right)^{p},$$

$$F_{a_0,b_m} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial a_0}\frac{\partial \mathbf{x}}{\partial b_m}\right] = \frac{8\pi^2|A|^2}{\sigma^2}\sum_{n\in\mathbb{N}}\sin\left(\phi_m^{\mathrm{FM}}(n)\right), \quad (15)$$

$$F_{a_0,\phi_m} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial a_0}\frac{\partial \mathbf{x}}{\partial \phi_m}\right] = \frac{8\pi^2|A|^2 b_m}{\sigma^2}\sum_{n\in\mathbb{N}}\cos\left(\phi_m^{\mathrm{FM}}(n)\right), \quad (16)$$

$$F_{b_m,\phi_{m'}} = \frac{2}{\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H}{\partial b_m}\frac{\partial \mathbf{x}}{\partial \phi_{m'}}\right], \quad (17)$$
$$= \frac{8\pi^2|A|^2 b_{m'}}{\sigma^2}\sum_{n\in\mathbb{N}}\sin\left(\phi_m^{\mathrm{FM}}(n)\right)\cos\left(\phi_{m'}^{\mathrm{FM}}(n)\right),$$

where $\mathbb{N} = [n_0,\cdots,n_0+N-1]$.

With the above equations, we can compute the exact FIM of dimension $(P+2M+1)\times(P+2M+1)$. Since we are more interested in the PPS parameters $\{a_p\}_{p=1}^{P}$, we group the unknown parameters in the following way:

$$\boldsymbol{\theta} = [\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T]^T,$$
$$\boldsymbol{\alpha} = [Na_1,\cdots,N^P a_P]^T, \quad \boldsymbol{\beta} = [a_0, \mathbf{b}^T, \boldsymbol{\phi}^T]^T. \quad (18)$$

with $\mathbf{b} = [b_1,\cdots,b_M]^T$ and $\boldsymbol{\phi} = [\phi_1,\cdots,\phi_M]^T$. As a result, the overall FIM is partitioned into blocks:

$$\mathbf{F}(\boldsymbol{\theta}) = \begin{bmatrix}\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} & \mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\\ \mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^T & \mathbf{F}_{\boldsymbol{\beta},\boldsymbol{\beta},}\end{bmatrix}, \quad (19)$$

where $\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}$ is a $P\times P$ square matrix with elements given by

$$[\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}]_{p,q} = F_{a_p,a_q}, \quad p,q=1,2,\cdots,P, \quad (20)$$

in (8), $\mathbf{F}_{\boldsymbol{\beta},\boldsymbol{\beta}}$ is a $(2M+1)\times(2M+1)$ square matrix with elements given by

$$\mathbf{F}_{\boldsymbol{\beta},\boldsymbol{\beta}} = \begin{bmatrix}F_{a_0,a_0} & \mathbf{F}_{a_0,\mathbf{b}} & \mathbf{F}_{a_0,\boldsymbol{\phi}}\\ \mathbf{F}_{a_0,\mathbf{b}}^T & \mathbf{F}_{\mathbf{b},\mathbf{b}} & \mathbf{F}_{\mathbf{b},\boldsymbol{\phi}}\\ \mathbf{F}_{a_0,\boldsymbol{\phi}}^T & \mathbf{F}_{\mathbf{b},\boldsymbol{\phi}}^T & \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\phi}}\end{bmatrix}, \quad (21)$$

with $F_{a_0,a_0}, [\mathbf{F}_{\mathbf{b},\mathbf{b}}]_{m,m'}, [\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\phi}}]_{m,m'}$ given by (9), (10) and (11), and $[\mathbf{F}_{a_0,\mathbf{b}}]_m, [\mathbf{F}_{a_0,\boldsymbol{\phi}}]_m, [\mathbf{F}_{\mathbf{b},\boldsymbol{\phi}}]_{m,m'}$ in (15), (16) and (17), respectively. And $\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a $P\times(2M+1)$ matrix

$$\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \begin{bmatrix}F_{a_1,a_0} & \mathbf{F}_{a_1,\mathbf{b}} & \mathbf{F}_{a_1,\boldsymbol{\phi}}\\ \vdots & \vdots & \vdots\\ F_{a_P,a_0} & \mathbf{F}_{a_P,\mathbf{b}} & \mathbf{F}_{a_P,\boldsymbol{\phi}}\end{bmatrix}, \quad (22)$$

with $F_{a_p,a_0}, [\mathbf{F}_{a_p,\mathbf{b}}]_m$ and $[\mathbf{F}_{a_p,\boldsymbol{\phi}}]_m$ given by (12), (13) and (14), respectively. With the block matrix inversion lemma, the CRBs for estimating the PPS parameters are given by

$$\mathrm{cov}(\hat{a}_p) \geq \mathrm{CRB}(\hat{a}_p) = \left[\left(\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\alpha}} - \mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbf{F}_{\boldsymbol{\beta},\boldsymbol{\beta}}^{-1}\mathbf{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^T\right)^{-1}\right]_{p,p}/N^{2p},$$

where the denominator of N^{2p} is to scale the CRBs for $\boldsymbol{\alpha} = [Na_1,\cdots,N^P a_P]^T$ back to a_p.

Remarks: It is clear that the derived CRBs are a function of the SNR (i.e., $|A|^2/\sigma^2$), the sampling indices $[n_0,\cdots,n_0+N-1]$, and the signal parameters including the PPS phase parameters $\{a_p\}_{p=0}^{P}$ and the sinusoidal FM parameters b_m and ϕ_m. Specifically, the FIM computations from (8) to (17) involve the PPS phase parameters $\{a_p\}_{p=0}^{P}$ via the phase function $\phi_m^{\mathrm{FM}}(n) = 2\pi m c_0\sum_{p=1}^{P} a_p n^p/p! + \phi_m$ of the m-th sinusoidal FM component. It is worthy noting that such an involvement of $\{a_p\}_{p=0}^{P}$ in the computation of CRBs is *unique* for the coupled mixture model. This effect has *not* been observed in the literature for the cases of the pure PPS [2] and the independent mixture signal [26]. In both cases, the CRBs are not a function of the PPS parameters.

B. Joint CRBs for A and σ^2

When the amplitude A and noise variance σ^2 are unknown, the joint CRB is shown to be decoupled, i.e., the cross-FIM matrices $\mathbf{F}_{\boldsymbol{\theta},A} = \mathbf{0}$ and $\mathbf{F}_{\boldsymbol{\theta},\sigma^2} = \mathbf{0}$. And we have $F_{A,A} = 2N/\sigma^2$ and $F_{\sigma^2,\sigma^2} = N/\sigma^4$ which leads to $\mathrm{CRB}(\hat{A}) = \sigma^2/(2N)$ and $\mathrm{CRB}(\hat{\sigma}^2) = \sigma^4/N$.

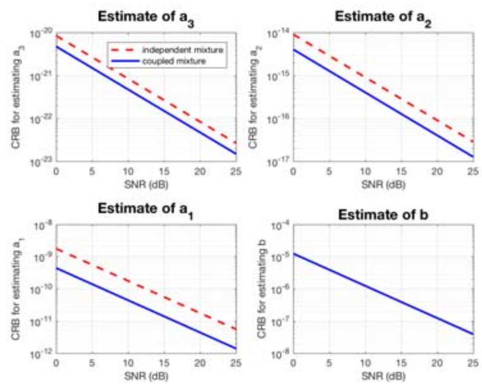

Fig. 2. The CRB as a function of SNR for the independent and coupled mixture signals with $M = 1$, $P = 3$, and $N = 2048$.

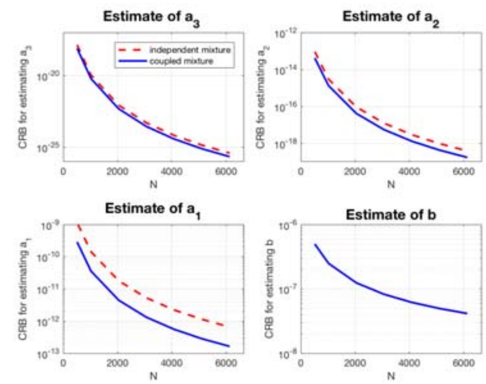

Fig. 3. The CRBs versus the number of samples for the independent and coupled mixture signals with $M = 1$, $P = 3$, and SNR $= 20$ dB.

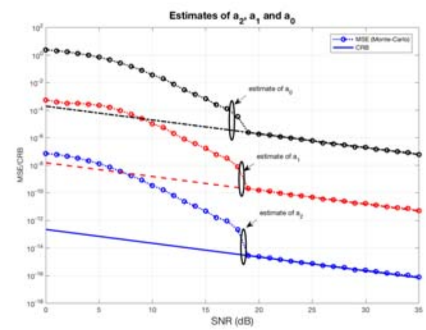

Fig. 4. The CRBs are achievable by a phase unwrapping estimator for estimating a_2, a_1 and a_0 of a coupled mixture signal with $M = 1$ and $P = 2$ at high SNRs when $N = 512$.

C. Connection to CRBs for Pure PPS

When $b_m = 0$ and $\phi_m^{\mathrm{FM}}(n) = 0$, the coupled mixture model reduces to the pure PPS model. In this case, the parameter set includes only the PPS phase parameters $\boldsymbol{\theta} = [\mathbf{a}^T, \mathbf{b}^T]^T$ with $\mathbf{a} = [a_1, \cdots, a_P]^T$ and $\mathbf{b} = [a_0]$, and the FIM entries from (8) to (17) are no longer a function of the PPS parameters $\{a_p\}_{p=1}^{P}$. Specifically, we have

$$F_{a_p,a_q} = \frac{8\pi^2|A|^2}{p!q!\sigma^2} \sum_{n=n_0}^{n_0+N-1} \left(\frac{n}{N}\right)^{p+q},$$

$$F_{a_0,a_0} = \frac{8\pi^2|A|^2}{\sigma^2}N, \quad F_{a_p,a_0} = \frac{8\pi^2|A|^2}{p!\sigma^2} \sum_{n=n_0}^{n_0+N-1} \left(\frac{n}{N}\right)^{p},$$

which agrees with the exact FIMs for the pure PPS when $n_0 = 0$ [2, Eqs. (13)-(15)] and when $n_0 = -(N-1)/2$ [5, Eq. (3)]. As a result, when $b_m = 0$, the derived CRBs in this paper reduce to the corresponding ones for the pure PPS.

IV. Numerical Results

In the following, we compare the derived CRBs with the counterparts for the independent mixture signal and show the achievability of the derived CRBs.

A. Comparison to CRBs for The Independent Mixture Signal

We consider a mixture signal of a single sinusoidal FM component ($M = 1$) and a PPS with order $P = 3$. Other parameters are $A = 1$, $a_0 = 0$, $a_1 = 0.15$, $a_2 = 1.3889 \cdot 10^{-4}$, $a_3 = 1.3022 \cdot 10^{-6}$, $b_1 = 0.05$, $\phi_1 = 0$, $c_0 = 0.1$ and $N = 2048$. For the coupled mixture signal of (3), the sinusoidal FM frequency is determined by $f_0(a_1, a_2, a_3) = c_0(a_1 + a_2 n/2 + a_3 n^2/6)$, while for the independent mixture signal f_0 is chosen to be 0.0676. Fig. 2 compares the CRBs as a function of SNR for the two mixture models. It is seen that, for estimating the PPS parameters $\{a_1, a_2, a_3\}$, the CRBs for the coupled mixture of (3) (blue curves) are smaller than those for the independent mixture of (5) (red curves), which suggests that extracting the information of a_p from both the PPS and FM components may lead to better estimation accuracy than conventional approaches that rely on the PPS component only. For estimating the sinusoidal FM index b, the two CRBs are very close to each other. Note that the CRBs for estimating $\{a_p\}$ in Fig. 2 can be translated to the CRBs for estimating *physical* parameters, e.g., the initial velocity v_0 and acceleration a in (4), by applying a proper scaling factor of $h^2/(\Delta T)^{2p}$.

Then, we fix SNR $= 20$ dB and vary the number of samples N with $n_0 = 0$. As shown in Fig. 3, the CRBs become smaller as N is larger. Similarly, the CRBs for estimating a_p for the coupled mixture of (3) are smaller than the counterparts for the independent mixture of (5).

B. The Achievability of CRBs

The maximum likelihood estimate (MLE) is known to achieve the CRB asymptotically, but it involves a $(2M + P)$-dimensional search and, hence, it is infeasible to numerically evaluate its mean squared error (MSE). Instead, we implement a simple phase unwrapping estimator. Fig. 4 shows, with 500 Monte-Carlo runs, the measured MSEs for estimating the PPS phase parameters $\{a_0, a_1, a_2\}$ of a coupled mixture with $M = 1$ and $P = 2$ when $N = 512$. It is seen that the measured MSEs achieve the corresponding CRBs when SNR ≥ 19 dB.

V. Conclusion

In this paper, we have introduced a new coupled mixture model of the PPS and sinusoidal FM signals. The coupling effect is seen from the fact that the sinusoidal FM frequency is a function of the PPS phase parameters. Given the new model, we have derived the CRBs for estimating unknown parameters. It has been found that, due to the coupling effect, the derived CRBs are a function of the PPS phase parameters, which is different from the cases of the pure PPS and the independent mixture signal.

References

[1] S. Peleg and B. Porat, "Estimation and classification of signals with polynomial phase," *IEEE Transactions on Information Theory*, vol. 37, no. 2, pp. 422–430, 1991.

[2] S. Peleg and B. Porat, "The Cramér-Rao lower bound for signals with constant amplitude and polynomial phase," *IEEE Trans. on Signal Processing*, vol. 39, no. 3, pp. 749–752, March 1991.

[3] S. Peleg and B. Friedlander, "The discrete polynomial-phase transform," *IEEE Trans. on Signal Processing*, vol. 43, no. 8, pp. 1901–1914, Aug. 1995.

[4] S. Barbarossa and V. Petrone, "Analysis of polynomial-phase signals by the integrated generalized ambiguity function," *IEEE Trans. on Signal Processing*, vol. 45, no. 2, pp. 316–327, Feb. 1997.

[5] B. Ristic and B. Boashash, "Comments on "the Cramér-Rao lower bounds for signals with constant amplitude and polynomial phase"," *IEEE Trans. on Signal Processing*, vol. 46, no. 6, pp. 1708–1709, June 1998.

[6] P. O'Shea, "A new technique for instantaneous frequency rate estimation," *IEEE Signal Processing Letters*, vol. 9, no. 8, pp. 251–252, Aug. 2002.

[7] P. O'Shea, "A fast algorithm for estimating the parameters of a quadratic FM signal," *IEEE Trans. on Signal Processing*, vol. 52, no. 2, pp. 385–393, Feb. 2004.

[8] M. Farquharson and P. O'Shea, "Extending the performance of the cubic phase function algorithm," *IEEE Trans. on Signal Processing*, vol. 55, no. 10, pp. 4767–4774, Oct. 2007.

[9] D. S. Pham and A. M. Zoubir, "Analysis of multicomponent polynomial phase signals," *IEEE Trans. on Signal Processing*, vol. 55, no. 1, pp. 56–65, Jan 2007.

[10] P. Wang, I. Djurovic, and J. Yang, "Generalized high-order phase function for parameter estimation of polynomial phase signal," *IEEE Trans. on Signal Processing*, vol. 56, no. 7, pp. 3023–3028, July 2008.

[11] P. Wang, H. Li, I. Djurović, and B. Himed, "Instantaneous frequency rate estimation for high-order polynomial-phase signals," *IEEE Signal Processing Letters*, vol. 16, no. 9, pp. 782–785, September 2009.

[12] R. G. McKilliam and I. V. L. Clarkson, "Identifiability and aliasing in polynomial-phase signals," *IEEE Trans. on Signal Processing*, vol. 57, no. 11, pp. 4554–4557, Nov 2009.

[13] A. Amar, A. Leshem, and A. J. van der Veen, "A low complexity blind estimator of narrowband polynomial phase signals," *IEEE Trans. on Signal Processing*, vol. 58, no. 9, pp. 4674–4683, Sep 2010.

[14] P. O'Shea, "On refining polynomial phase signal parameter estimates," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 46, no. 3, pp. 978–987, July 2010.

[15] I. Djurovic, M. Simeunovic, S. Djukanovic, and P. Wang, "A hybrid CPF-HAF estimation of polynomial-phase signals: Detailed statistical analysis," *IEEE Trans. on Signal Processing*, vol. 60, no. 10, pp. 5010–5023, Oct. 2012.

[16] R. G. McKilliam, B. G. Quinn, I. V. L. Clarkson, B. Moran, and B. N. Vellambi, "Polynomial phase estimation by least squares phase unwrapping," *IEEE Trans. on Signal Processing*, vol. 62, no. 8, pp. 1962–1975, April 2014.

[17] I. Djurovic and L. J. Stankovic, "Quasi maximum likelihood estimator of polynomial phase signals," *IET Signal Processing*, vol. 13, no. 4, pp. 347–359, June 2014.

[18] I. Djurovic and M. Simeunovic, "Resolving aliasing effect in the QML estimation of PPSs," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 52, no. 3, pp. 1494–1499, June 2016.

[19] I. Djurovic, "QML-RANSAC: PPS and FM signals estimation in heavy noise environments," *Signal Processing*, vol. 130, pp. 142–151, Jan. 2017.

[20] S. Kar, P. K. Varshney, and M. Palaniswami, "Cramér-Rao bounds for polynomial signal estimation using sensors with AR(1) drift," *IEEE Trans. on Signal Processing*, vol. 60, no. 10, pp. 5494–5507, October 2012.

[21] S.-R. Huang, R. M. Lerner, and K. J. Parker, "On estimating the amplitude of harmonic vibration from the Doppler spectrum of reflected signals," *J. Acoust. Soc. Amer.*, vol. 88, pp. 2702–2712, Dec. 1990.

[22] J.-E. Wilbur and R. J. McDonald, "Nonlinear analysis of cyclically correlated spectral spreading in modulated signals," *J. Acoust. Soc. Amer.*, vol. 92, pp. 219–230, July 1992.

[23] M. R. Bell and R. A. Grubbs, "JEM modeling and measurement for radar target identification," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, pp. 73–87, Jan. 1993.

[24] S. Palumbo, S. Barbarossa, A. Farina, and M. R. Toma, "Classification techniques of radar signals backscattered by helicopter blades," in *Proceedings of Int. Symp. Digital Signal Process.*, London, UK, July 1996.

[25] F. Gini and G. B. Giannakis, "Parameter estimation of hybrid hyperbolic FM and polynomial phase signals using the multi-lag high-order ambiguity function," in *Proceedings of The Thirty-First Asilomar Conference on Signals, Systems, and Computers*, Nov. 1997, vol. 1, pp. 250–254.

[26] F. Gini and G. B. Giannakis, "Hybrid FM-polynomial phase signal modeling: Parameter estimation and Cramér-Rao bounds," *IEEE Trans. on Signal Processing*, vol. 47, no. 2, pp. 363–377, Feb. 1999.

[27] O. Besson, G. B. Giannakis, and F. Gini, "Improved estimation of hyperbolic frequency modulated chirp signals," *IEEE Trans. on Signal Processing*, vol. 47, no. 5, pp. 1384–1388, May 1999.

[28] J. F. Gieras, Z. J. Piech, and B. Tomczuk, *Linear Synchronous Motors: Transportation and Automation Systems*, CRC Press, Boca Raton, FL, 2011.

[29] B. M. Wilamowski and J. D. Irwin, *The Industrial Electronics Handbook*, CRC Press, Boca Raton, FL, 2011.

[30] S. M. Kay, *Modern Spectral Estimation: Theory and Application*, Prentice Hall, Englewood Cliffs, NJ, 1988.