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Claus Danielson, Leila Bridgeman, and Stefano Di Cairano

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I. Introduction

Many industrial control problems involve systems whose dynamics and constraints switch between distinct modes of operation that can be modeled as switched constrained systems. A switched system is a family of dynamic systems with a switching signal specifying which of the dynamic modes is active as a function of time [1]. Switched constrained systems are switched systems subject to mode dependent constraints on the state and input. Switched constrained linear systems are used in a variety of disciplines, including automotive applications, where driveline dynamics evolve through distinct modes during gearshifts [2] and in heating, ventilation, and air conditioning of buildings, where the heating or cooling to a zone may be engaged or disengaged [3], changing the overall system structure. Switched linear models are also commonly used for modeling the dynamics of walking [4], [5].

This paper seeks to guarantee constraint satisfaction for switched constrained linear systems. We consider switching signals that force the system to dwell in each mode for a minimum length of time, called the dwell-time. Dwell-time is an important concept for analyzing switched systems because a sufficiently long dwell-time allows the overall switched system to inherit stability from the stability of its individual modes [1]. For stable systems, there are additional conditions that can guarantee constraint satisfaction in a neighborhood of the stabilized equilibrium [6], however this neighborhood is typically overly conservative.

Switched linear systems are a special case of polytopic linear parameter varying (pLPV) systems and therefore many of the analysis and design techniques for pLPV systems can be applied to switched linear systems [7]–[11]. In [11] the authors computed invariant sets for system dynamics evolving inside a polytopic set of dynamic systems, albeit with constraints that are not time-varying. Regardless, the use of polytopic linear parameter systems may be unnecessarily conservative since all dynamics in the polytope are considered and no restriction is placed on how often the dynamics change, effectively requiring controllers that ensure stability and constraint satisfaction for arbitrary switching signals.

In this paper, we derive mode dependent control (positive) invariant sets with the additional property that, when the mode changes, it is possible to transition between the invariant sets within the dwell-time without violating constraints. If the initial state of the system can reach the current mode's invariant set during the dwell-time, then it is possible to ensure constraint satisfaction for any admissible switching signal. These control (positive) invariant sets can be used to design recursively feasible model predictive controllers. Model predictive control (MPC) has been applied to switched linear systems in recent years [12]–[14]. Most recently, [15] proposed a discrete-time linear MPC that used a finite preview of the switching signal to guarantee stability and recursive feasibility. However, in that paper the constraints are time-invariant. Even with time-invariant constraints, the invariant sets presented in [15] is overly conservative. In contrast, this paper considers the more general case where both the dynamics and constraints can switch modes. Furthermore, the invariant sets derived here do not require a preview of the switching signal and are not conservative in the sense that constraint satisfaction can be guaranteed if and only if the initial condition can reach the invariant sets within the dwell-time.

The paper is organized as follows. In Section II, we formally define switched constrained linear systems and the constraint satisfaction problem and provide sufficient conditions for constraint satisfaction. In Section III, we propose an algorithm to compute invariant sets that satisfy the sufficient conditions. Furthermore, we show that the invariant sets are maximal in the sense that constraint satisfaction is guaranteed if and only if the initial state of the system can reach these sets within the dwell-time. In Section IV, we show how these sets can be used to design recursively feasible model predictive controllers. Finally, in section V we apply the theory developed in this paper to a vehicle lane-changing case study.

1) Definitions: Consider the autonomous system $x^+ = f(x)$. A set $\mathcal{O}$ is positive invariant if $f(x) \in \mathcal{O}$ for all $x \in \mathcal{O}$. A necessary and sufficient condition for positive invariance is $\mathcal{O} \subseteq \mathrm{Pre}(\mathcal{O})$ where $\mathrm{Pre}(\Omega) = \{x : f(x) \in \Omega\}$. Similarly for the system $x^+ = f(x,u)$, a set $\mathcal{C}$ is control invariant if for all $x \in \mathcal{C}$, there exists $u \in \mathcal{U}$ such that $f(x,u) \in \mathcal{C}$. A necessary and sufficient condition for control invariance is $\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C})$ where $\mathrm{Pre}(\Omega) = \{x : \exists u, f(x,u) \in \Omega\}$.

II. Problem Statement

This paper studies the control of constrained linear systems whose dynamics and constraints change between a finite set of modes over time. We consider the following switched constrained linear system

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1a}$$

$$x(t) \in \mathcal{X}_{\sigma(t)} \tag{1b}$$

$$u(t) \in \mathcal{U}_{\sigma(t)} \tag{1c}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state and $u(t) \in \mathbb{R}^{n_u^{\sigma(t)}}$ is the input. The number of inputs, n_u^i, may depend on the mode $i \in \mathbb{I}$. The switching signal, $\sigma : \mathbb{N} \to \mathbb{I}$, is an unknown exogenous input that switches the dynamics $A_i \in \mathbb{R}^{n_x \times n_x}$ and $B_i \in \mathbb{R}^{n_x \times n_u^i}$, and the constraint sets $\mathcal{X}_i \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U}_i \subseteq \mathbb{R}^{n_u^i}$ between a finite number $|\mathbb{I}| < \infty$ of modes $\mathbb{I} \subset \mathbb{N}$. We assume that the pair (A_i, B_i) is controllable and that the sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are full-dimensional and contain an equilibrium of the dynamics (1a) in their interior for each mode $i \in \mathbb{I}$.

The discrete-times $t_s \in \mathbb{N}$ at which the switching signal $\sigma : \mathbb{N} \to \mathbb{I}$ changes mode $\sigma(t_s) \neq \sigma(t_s - 1)$ are called switching instants. The system (1) is initialized after a mode switch t_0 and the future switch times are formally defined as $t_{s+1} = \min\{t \geq t_s : \sigma(t) \neq \sigma(t_s)\}$. If the signal σ only switches a finite-number of times $\bar{s} < \infty$ then, for mathematical convenience, we define $t_s = \infty$ for all $s > \bar{s}$ so that $\mathbb{N} = \bigcup_{s=0}^{\infty}[t_s, t_{s+1}]$. The dwell-time $\mathrm{dwell}(\sigma)$ of a switching signal $\sigma : \mathbb{N} \to \mathbb{I}$ is the minimal length of time that the switching signal dwells in each mode, $\mathrm{dwell}(\sigma) = \min\{t_{s+1} - t_s : s \in \mathbb{N}\}$. The set of switching signals with dwell-times of at least d time-steps is denoted by

$$\Sigma_d = \{\sigma : \mathbb{N} \to \mathbb{I} : \mathrm{dwell}(\sigma) \geq d\}. \tag{2}$$

For a given switching signal $\sigma \in \Sigma_d$, a feasible solution for the system (1) is a pair of state $\{x(t)\}_{t=t_0}^{\infty}$ and input $\{u(t)\}_{t=t_0}^{\infty}$ trajectories that satisfies the mode dependent dynamics (1a) and constraints (1b) and (1c). The existence of a feasible solution depends on the initial state $x(t_0)$ and the future switching sequence $\sigma(t)$ for $t \geq t_0$, but not on the initial time t_0 since t_0 was a switching time. We are interested in characterizing the set of initial states $x(t_0)$ for which the system (1) has feasible solutions for all switching signals $\sigma \in \Sigma_d$ with dwell-time $\mathrm{dwell}(\sigma) \geq d$. Note that the set of feasible initial states $x(t_0)$ is dependent on the initial mode $\sigma(t_0)$. This problem is formally stated below.

Problem 1: For each mode $i \in \mathbb{I}$, find a set $\mathcal{X}_i^0 \subseteq \mathcal{X}_i$ such that for any initial state $x(t_0) \in \mathcal{X}_i^0$ and admissible switching sequence $\sigma(t) \in \Sigma_d$ beginning in mode $i = \sigma(t_0) \in \mathbb{I}$, there exists a sequence of control inputs $u(t)$ satisfying (1c) such that the sequence of states produced by the dynamics (1a) satisfies the state constraints (1b) for all time $t \geq t_0 \in \mathbb{N}$. □

Remark 1: In many applications, the set of switching signals can be further restricted because only switches between certain pairs of modes are permitted. For example, if $\sigma(t_s) = i$ then $\sigma(t_{s+1}) \neq j$. The allowable mode switches can be specified by a directed graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ where the graph nodes $\mathbb{I}$ are the modes of the switched system (1) and each directed edge $(i,j) \in \mathbb{E}$ indicates that a switch from mode $\sigma(t_s) = i$ to mode $\sigma(t_{s+1}) = j$ is allowed. The set of switching signals, $\sigma : \mathbb{N} \to \mathbb{I}$, that satisfy the dwell-time and mode change restrictions, $\mathrm{dwell}(\sigma) \geq d$ and $\mathbb{G}$, will be denoted by $\Sigma_d(\mathbb{G}) = \{\sigma : \mathbb{N} \to \mathbb{I} : \ \mathrm{dwell}(\sigma) \geq d, (\sigma(t_s), \sigma(t_{s+1})) \in \mathbb{E}, \ \forall s \in \mathbb{N}\}$.

The initial condition sets for which the switched system (1) is guaranteed to have a feasible solution for all switching signals in $\Sigma_d(\mathbb{G})$, denoted by $\mathcal{X}_i^0(\mathbb{G})$, are larger than the sets $\mathcal{X}_i^0$ in Problem 1 since the set of possible switching signals is more restrictive, i.e. $\mathcal{X}_i^0(\mathbb{G}) \supseteq \mathcal{X}_i^0$ because $\Sigma_d(\mathbb{G}) \subseteq \Sigma_d$. Throughout this paper, we will remark on how our results can be modified for switching signals in the more restrictive set $\Sigma_d(\mathbb{G})$. □

Our analysis of Problem 1 will use the predecessor-operator. For the controlled switched system (1) operating in a constant mode $i \in \mathbb{I}$, the k-step predecessor of a set $\Omega \subseteq \mathcal{X}_i$ is defined recursively by

$$\mathrm{Pre}_i^0(\Omega) = \Omega \tag{3a}$$

$$\mathrm{Pre}_i^{k+1}(\Omega) = \{x \in \mathcal{X}_i : \exists u \in \mathcal{U}_i \text{ s.t. } A_i x + B_i u \in \mathrm{Pre}_i^k(\Omega)\} \tag{3b}$$

for $k \in \mathbb{N}$. The set $\mathrm{Pre}_i^k(\Omega) \subseteq \mathcal{X}_i$ is the set of states $x \in \mathcal{X}_i$ that can be mapped, under the dynamics of mode $i \in \mathbb{I}$, into the set Ω in k discrete-time instants without violating the state $\mathcal{X}_i$ and input constraints $\mathcal{U}_i$ of mode $i \in \mathbb{I}$.

The following theorem gives a pair of sufficient conditions on the initial state ensuring that the system (1) has a feasible solution for any admissible switching signal $\sigma \in \Sigma_d$.

Theorem 1: Consider a collection of sets $\mathcal{C}_i \subseteq \mathcal{X}_i$ for $i \in \mathbb{I}$ that satisfy the conditions (1) $\mathcal{C}_i$ is control invariant with respect to the dynamics of mode $i \in \mathbb{I}$, and (2) $\mathcal{C}_i$ is d-step reachable from $\mathcal{C}_j$ under the dynamics of mode $i \in \mathbb{I}$ for all $j \in \mathbb{I}$. Then for any admissible switching signal $\sigma \in \Sigma_d$, if the initial state satisfies $x(t_0) \in \mathcal{X}_i^0 = \mathrm{Pre}_i^d(\mathcal{C}_i)$ where $i = \sigma(t_0) \in \mathbb{I}$ is the initial mode, then the switched constrained linear system (1) has a feasible solution.

Remark 2: For switching signals $\sigma \in \Sigma_d(\mathbb{G})$ restricted to the smaller set $\Sigma_d(\mathbb{G})$, Condition (2) of Theorem 1 only needs to hold for admissible switches $(j,i) \in \mathbb{E}$, i.e. the invariant set $\mathcal{C}_i$ of mode $i \in \mathbb{I}$ only needs to be reachable from the invariant set $\mathcal{C}_j$ of mode $j \in \mathbb{I}$ if it is possible $(j,i) \in \mathbb{E}$ to switch to mode $i \in \mathbb{I}$ from mode $j \in \mathbb{I}$. □

Each set $\mathcal{C}_i$ is control invariant for the system (1) if the mode is constant $\sigma(t) = i$ for all $t \in \mathbb{N}$. However, the individual sets $\mathcal{C}_i$ are not invariant when the system (1) changes modes, $\sigma(t{+}1) \neq \sigma(t)$, since the state $x(t)$ may leave the set $\mathcal{C}_i$ under the dynamics of the next mode $\sigma(t{+}1) \in \mathbb{I}$. Collectively, however, the sets $\mathcal{C}_i$ for $i \in \mathbb{I}$ are invariant in the sense that for any state $x(t) \in \mathcal{C}_{\sigma(t)}$ and any possible future mode, $\sigma(t+1) \in \mathbb{I}$, there exists a control input $u(t) \in \mathcal{U}_{\sigma(t)}$ such that the future state $A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \in \mathrm{Pre}^d_{\sigma(t+1)}\big(\mathcal{C}_{\sigma(t+1)}\big)$ can reach the control invariant set $\mathcal{C}_{\sigma(t+1)}$ of mode $\sigma(t+1)$ without violating constraints. Since the collection of sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ is invariant under switching signals $\sigma \in \Sigma_d$, we call these sets *control switch-invariant sets*. We call the collection of control switch-invariant sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ *maximal* if $\mathcal{X}_i^0 = \mathrm{Pre}_i^d(\mathcal{C}_i)$ contains every initial condition for which it is possible to satisfy constraints. For now we will postpone the question of whether a collection of control switch-invariant sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ can be maximal. This question will be answered, in the affirmative, in Section III, where we provide an algorithm for computing the collection of maximal control switch-invariant sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$.

Mode switches are a powerful disturbance on the system. Therefore, the control switch-invariant sets $\mathcal{C}_i$ must be designed more conservatively than traditional invariant sets to ensure that a mode switch does not produce a constraint violation. During the period $t = t_s, \dots, t_s+d$ immediately after a mode switch, the system will not be disturbed by another mode switch due to the dwell-time restrictions $\sigma \in \Sigma_d$. Thus, the initial condition sets $\mathcal{X}_i^0 = \mathrm{Pre}_i^d(\mathcal{C}_i)$ are less conservative than the control switch-invariant sets $\mathcal{C}_i \subseteq \mathcal{X}_i^0$ since they will not encounter a mode switch for the next d time-steps.

We call a collection of control switch-invariant sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ full-dimensional if the individual sets $\mathcal{C}_i \subseteq \mathbb{R}^n$ have non-empty interiors and lower-dimensional otherwise. If the sets $\mathcal{X}_i$ and $\mathcal{U}_i$ contains the origin for each mode $i \in \mathbb{I}$, then the system (1) always has a collection of lower-dimensional control switch-invariant sets, namely the origin $\mathcal{C}_i = \{0\}$ for each mode $i \in \mathbb{I}$. In Section III we will provide an algorithm for computing full-dimensional switch-invariant sets.

Note that the sequence of control inputs $u(t) \in \mathcal{U}_{\sigma(t)}$ needed to keep the state $x(t+1) \in \mathcal{C}_{\sigma(t)}$ in the control switch-invariant sets $\mathcal{C}_{\sigma(t)}$ depends on the current state $x(t)$ and mode $\sigma(t)$, as well as the amount of time, t–t_s, since the last switch. Thus, the controller $u(t) = \kappa(x(t), \sigma(t), t-t_s)$ needed to satisfy the state and input constraints (1b) and (1c), is in general time-varying.

A. Closed-loop Constraint Satisfaction

In this section we consider a special case of Problem 1 where the system (1) is being controlled by a given mode and time dependent controller

$$u(t) = \kappa\big(x(t), \sigma(t), t-t_s\big) \tag{4}$$

where the controller (4) is time-varying for the first d time-steps $t = [t_s, \dots, t_s+d-1]$ after a mode switch $\sigma(t_s) \neq \sigma(t_s-1)$, and time-invariant until the next mode switch, i.e. $\kappa(\cdot,\cdot,t-t_s) = \kappa(\cdot,\cdot,d)$ for $t \in [t_s+d, \dots, t_{s+1}]$. The time-varying portion of the controller (4) is used to reach a set of states where it is safe to change modes and the time-invariant portion of the controller is used to keep the state in this set.

The system (1) in closed-loop with the time-varying, mode-dependent controller (4) is given by

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}\kappa(x(t), \sigma(t), t-t_s) \tag{5a}$$
$$x(t) \in \mathcal{X}_{\sigma(t)} \tag{5b}$$
$$\kappa(x(t), \sigma(t), t-t_s) \in \mathcal{U}_{\sigma(t)}. \tag{5c}$$

For a given switching signal $\sigma \in \Sigma_d$, a feasible solution of the system (5) is a state trajectory $\{x(t)\}_{t=t_0}^\infty$ that satisfies the dynamics (5a), and constraints (5b) and (5c) for all $t \geq t_0 \in \mathbb{N}$. We would like to characterize the set of initial conditions for which the autonomous system (5) is guaranteed to produce a feasible solution for any admissible switching signal $\sigma \in \Sigma_d$.

Our analysis of (5) will again use the concept of a predecessor set. However, in this case, we must specify the time at which the predecessor set is initialized because the system (5) is time-varying. The set $\mathrm{Pre}_i^k(\Omega) \subseteq \mathcal{X}_i$ is the set of states $x(t_s+d-k) \in \mathcal{X}_i$ at time t_s+d-k that will reach Ω at time $t = t_s+d$ without violating state (5b) and input (5c) constraints under the closed-loop dynamics (5a). The predecessor-set is initialized d time-steps after the most recent mode switch $t = t_s+d$ and propagated backward until the mode switch $t = t_s$. Thus, the predecessor-operator of a set Ω for the closed-loop switched system (5) operating in mode $\sigma(t_s) = i$ is defined recursively as

$$\mathrm{Pre}_i^0(\Omega) = \Omega \tag{6a}$$
$$\mathrm{Pre}_i^{k+1}(\Omega) = \big\{x \in \mathcal{X}_i : \kappa(x, i, d-k-1) \in \mathcal{U}_i \text{ and } A_i x + B_i\kappa(x, i, d-k-1) \in \mathrm{Pre}_i^k(\Omega)\big\} \tag{6b}$$

for $k = 0, \dots, d-1$.

Corollary 1 of Theorem 1 provides sufficient conditions on the initial state to guarantee that the autonomous system (5) produces feasible trajectories for all switching signals $\sigma \in \Sigma_d$.

Corollary 1: Consider a collection of sets $\mathcal{O}_i \subseteq \mathcal{X}_i$ for $i \in \mathbb{I}$ that satisfy (1) $\mathcal{O}_i$ is positive invariant for mode $i \in \mathbb{I}$ under the steady-state controller $u(t) = \kappa\big(x(t), i, d\big)$, and (2) $\mathcal{O}_i$ is reachable from $\mathcal{O}_j$ in d time-steps under the dynamics and constraints (5) of mode $i \in \mathbb{I}$ immediately after a mode switch. Then, if the initial state satisfies $x(t_0) \in \mathcal{X}_i^0 = \mathrm{Pre}_i^d(\mathcal{O}_i)$ where $i = \sigma(t_0) \in \mathbb{I}$ is the initial mode then the system (5) has a feasible solution for any admissible switching signals $\sigma \in \Sigma_d$.

Similarly to the control switch-invariant sets $\mathcal{C}_i$, the sets $\mathcal{O}_i$ are collectively positive invariant for the system (5) under admissible switching $\sigma \in \Sigma_d$ in the sense that for any state $x(t) \in \mathcal{O}_{\sigma(t)}$ in the positive invariant set $\mathcal{O}_{\sigma(t)}$ the future state $A_{\sigma(t)}x(t) + B_{\sigma(t)}\kappa(x(t), \sigma(t), t-t_s) \in \mathrm{Pre}^d_{\sigma(t+1)}\big(\mathcal{O}_{\sigma(t+1)}\big)$ can reach the positive invariant set $\mathcal{O}_j = \mathcal{O}_{\sigma(t+1)}$ of any possible future mode $\sigma(t+1) = j \in \mathbb{I}$. Thus, we call the collection of sets $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ *positive switch-invariant*. We call the collection of positive switch-invariant sets $\{\mathcal{O}_i^\infty\}_{i\in\mathbb{I}}$ *maximal* if $\mathrm{Pre}^d_{\sigma(t_0)}(\mathcal{O}^\infty_{\sigma(t_0)})$ contains every initial condition $x(t_0) \in \mathcal{X}_i^0 = \mathrm{Pre}^d_{\sigma(t_0)}(\mathcal{O}^\infty_{\sigma(t_0)})$ for which it is possible to satisfy constraints. Note that, unlike the control switch-invariant set, the sets $\mathrm{Pre}^d_{\sigma(t+1)}(\mathcal{O}_{\sigma(t+1)})$ are not necessarily positive invariant under the time-varying controller (4).

As with the controlled case, we call a collection of positive switch-invariant sets $\{\mathcal{O}_i\}_{i\in\mathbb{I}}$ full-dimensional if the individual sets $\mathcal{O}_i \subseteq \mathbb{R}^n$ have a non-empty interior and lower-dimensional otherwise. If each set $\mathcal{X}_i$ contains the origin, then the system (5) always has a collection of lower-dimensional positive switch-invariant sets, namely the origin $\mathcal{O}_i = \{0\}$ for each mode $i \in \mathbb{I}$.

III. Computing Maximal Switch-Invariant Sets

In this section we present algorithms for computing the maximal control and positive switch-invariant sets.

A. Computing Maximal Control Switch-Invariant Sets

A collection of control switch-invariant sets, $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$, can be computed using Algorithm 1. Algorithm 1 initializes the estimates $\{\Omega_i^k\}_{i\in\mathbb{I}}$ of the control switch-invariant sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ with the outer-approximations $\Omega_i^0 = \mathcal{X}_i \supseteq \mathcal{C}_i$ for each mode $i \in \mathbb{I}$. During each iteration, Algorithm 1 refines the outer estimates $\{\Omega_i^k\}_{i\in\mathbb{I}}$ by removing states $x \in \Omega_i^k$ that cannot be kept in the set Ω_i^k under the dynamics of mode $i \in \mathbb{I}$ and cannot reach Ω_j^k in d time-steps under the dynamics of mode $j \in \mathbb{I}$. This is accomplished by intersecting the sets Ω_i^k with the predecessor sets $\mathrm{Pre}_i(\Omega_i^k)$ and $\mathrm{Pre}_j^d(\Omega_j^k)$ where the predecessor-operator was defined in (3). The algorithm terminates when the estimates Ω_i^k of the control switch-invariant sets $\mathcal{C}_i$ have converged $\Omega_i^{k+1} = \Omega_i^k$ for each mode $i \in \mathbb{I}$.

Algorithm 1 Maximal control switch-invariant sets

1: **for** each mode $i \in \mathbb{I}$ **do**
2: $\quad \Omega_i^0 = \mathcal{X}_i$
3: **end for**
4: **repeat**
5: $\quad$ **for** each mode $i \in \mathbb{I}$ **do**
6: $\quad\quad \Omega_i^{k+1} = \Omega_i^k \cap \mathrm{Pre}_i(\Omega_i^k) \cap \big(\bigcap_{j\neq i} \mathrm{Pre}_j^d(\Omega_j^k)\big)$
7: $\quad$ **end for**
8: **until** $\Omega_i^{k+1} = \Omega_i^k$ for all $i \in \mathbb{I}$
9: $\mathcal{C}_i^\infty = \Omega_i^k$ for all $i \in \mathbb{I}$.

Algorithm 1 modifies the update rule $\Omega_i^{k+1} = \Omega_i^k \cap \mathrm{Pre}_i(\Omega_i^k)$ used to compute traditional control invariant sets [6]. The estimates Ω_i^k of the switch-invariant sets are updated by intersecting the update rule for traditional control invariant sets with each set of states, $\mathrm{Pre}_i^d(\Omega_j^k)$, that can reach the invariant sets Ω_j^k of mode $j \in \mathbb{I}$. This reflects the fact that the switch-invariant sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ must be more conservative to ensure that constraint violations do not occur during the transient after a mode change.

The iteration $k = k^\star$ for which Algorithm 1 terminates is called the determinedness index. The sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}} = \{\Omega_i^k\}_{i\in\mathbb{I}}$ are called finitely determined if the determinedness index is finite $k^\star < \infty$. The following theorem shows that Algorithm 1 produces the maximal control switch-invariant sets.

Theorem 2: Let the sets $\{C_i^\infty\}_{i\in\mathbb{I}}$ be finitely determined $k^\star < \infty$. Then the system (1) has a feasible solution for every admissible switching signal $\sigma \in \Sigma_d$ if and only if the initial state satisfies $x(t_0) \in \mathcal{X}^0_{\sigma(t_0)} = \mathrm{Pre}^d_{\sigma(t_0)}(\mathcal{C}^\infty_{\sigma(t_0)})$ where $\{C_i^\infty\}_{i\in\mathbb{I}}$ are the sets produced by Algorithm 1.

Remark 3: For switching signals in the restricted set $\sigma \in \Sigma_d(\mathbb{G})$ we only need to be able to reach the set $\mathcal{C}_j$ from $\mathcal{C}_i$ if a switch from mode $i \in \mathbb{I}$ to mode $j \in \mathbb{I}$ is allowed. That is, $\mathcal{C}_i \subseteq \mathrm{Pre}_j^d(\mathcal{C}_j)$ need not be imposed unless $(i,j) \in \mathbb{E}$. Thus, Algorithm 1 can be simplified by only intersecting Ω_i^k with reachable sets $\mathrm{Pre}_j^d(\Omega_j^k)$ for modes $j \in \mathbb{I}$ to which the system can switch $(j,i) \in \mathbb{E}$ producing the update rule $\Omega_i^{k+1} = \Omega_i^k \cap \mathrm{Pre}_i(\Omega_i^k) \cap \big(\bigcap_{(i,j)\in\mathbb{E}} \mathrm{Pre}_j^d(\Omega_j^k)\big)$. □

B. Computing Maximal Positive Switch-Invariant Sets

The maximal positive switch-invariant sets $\{\mathcal{O}_i^\infty\}_{i\in\mathbb{I}}$ can be computed using Algorithm 1 with the initialization

$$\Omega_i^0 = \mathcal{X}_i \cap \big\{x : \kappa(x,i,d) \in \mathcal{U}_i\big\}, \tag{7}$$

the predecessor-operator $\mathrm{Pre}_j^d(\Omega_j^k)$ was defined in (6), and the predecessor-operator without superscript $\mathrm{Pre}_i(\Omega_i^k)$ is given by $\mathrm{Pre}_i(\Omega_i^k) = \big\{x \in \mathcal{X}_i : \kappa(x,i,d) \in \mathcal{U}_i \text{ and } A_i x + B_i\kappa(x,i,d) \in \Omega_i^k\big\}$ where the controller $\kappa(x,i,k) = \kappa(x,i,d)$ for $k \geq d$ is used to render the set $\mathcal{O}_i$ positive invariant and the controller $\kappa(x,i,k)$ for $k = 0,\dots,d-1$ is used to reach the set $\mathcal{O}_i$. The following corollary shows that Algorithm 1 produces the maximal positive switch-invariant sets where $\mathcal{O}_i^\infty = \Omega_i^k$ for $i \in \mathbb{I}$.

Corollary 2: Let $\{O_i^\infty\}_{i\in\mathbb{I}}$ be finitely determined $k^\star < \infty$. Then constraints $\mathcal{X}_{\sigma(t)}$ and $\mathcal{U}_{\sigma(t)}$ can be satisfied for all admissible switching signals $\sigma \in \Sigma_d$ and all time $t \in \mathbb{N}$ if and only if the initial state satisfies $x(t_0) \in \mathrm{Pre}^d_{\sigma(t_0)}(\mathcal{O}^\infty_{\sigma(t_0)})$, where $\{O_i^\infty\}_{i\in\mathbb{I}}$ are the sets produced by Algorithm 1.

IV. MPC for Switched Linear Systems

In [15] we presented a model predictive controller (MPC) for constrained systems with switched dynamics which required knowledge of the future switching sequence $\sigma(t),\dots,\sigma(t+N)$ over the prediction horizon N. In this section we present a MPC where only the current mode $\sigma(t)$ of the system (1) and the dwell-time bound d are known.

The MPC computes the control input $u(t)$ by solving the following constrained finite-time optimal control problem

$$\begin{aligned}
\min \quad & p_{\sigma_{0|t}}(x_{N|t}) + \textstyle\sum_{k=0}^{N-1} q_{\sigma_{0|t}}(x_{k|t}, u_{k|t}) && \text{(8a)}\\
\text{s.t.} \quad & x_{k+1|t} = A_{\sigma_{0|t}} x_{k|t} + B_{\sigma_{0|t}} u_{k|t} && \text{(8b)}\\
& x_{k+1|t} \in \mathcal{X}_{\sigma_{0|t}},\ u_{k|t} \in \mathcal{U}_{\sigma_{0|t}} && \text{(8c)}\\
& x_{k+1|t} \in \mathcal{T}_{\sigma_{0|t}} \text{ for } t+k \geq t_s+d && \text{(8d)}
\end{aligned}$$

where $x_{0|t} = x(t)$ is the current state of the system (1), $x_{k|t}$ is the predicted state of the system under the control actions $u_{k|t}$ over the prediction horizon $N \geq d$, $\sigma_{0|t} = \sigma(t)$ is the current mode of the system, $\mathcal{T}_{\sigma(0|t)}$ is the terminal constraint, and t_s is the most recent mode switch instance. Since this paper is focused on constraint satisfaction, the terminal $p_{\sigma_{0|t}}(\cdot)$ and stage $q_{\sigma_{0|t}}(\cdot,\cdot)$ costs are unrestricted and can be selected to satisfy secondary control objectives such as stability or reference tracking for the individual modes. The optimal control problem (8) is solved assuming that the mode $\sigma(t) \in \mathbb{I}$ is constant $\sigma(k|t) = \sigma(0|t)$ over the horizon $k = 0, \ldots, N$.

The terminal constraint (8d) is applied for every time index $t+k \geq t_s+d$ after the dwell-time d has expired. This ensures that the predicted state $x_{k|t}$ enters the terminal region $\mathcal{T}_{\sigma_{0|t}}$ by the time the dwell-time expires and remains there until the next mode switch. The terminal constraint (8d) naturally loosens the constraints immediately after a mode switch and tightens the constraints as the dwell-time dwindles.

The MPC control input is the first element $u^\star_{0|t}$ of the optimal open-loop input sequence $u^\star_{0|t}, \ldots, u^\star_{N_t-1|t}$

$$u(t) = u^\star_{0|t}\big(x(t), \sigma(t), t-t_s\big). \tag{9}$$

The domain $\mathcal{D}_i(t–t_s) \subseteq \mathcal{X}_i$ of the model predictive controller (9) is the set of initial states $x_{0|t} = x(t) \in \mathcal{D}_i(t-t_s)$ for which the constrained finite-time optimal control problem (8) has a feasible solution. The domain of the model predictive controller (9) is time and mode dependent since it may be possible to satisfy constraints for an initial state $x_{0|t}$ under the dynamics and constraints of mode $i \in \mathbb{I}$ but not under those of mode $j \in \mathbb{I}$.

A. *Feasibility of the Switched MPC*

In this section we examine when the model predictive controller (9) satisfies the state and input constraints. Theorem 3 shows that the model predictive controller is recursively feasible when the terminal set $\mathcal{T}_{\sigma_{0|t}}$ is a control switch-invariant set $\mathcal{C}_{\sigma_{0|t}}$ or positive switch-invariant set $\mathcal{O}_{\sigma_{0|t}}$.

Theorem 3: Let $\mathcal{T}_{\sigma_{0|t}} = C_{\sigma_{0|t}}$ or $\mathcal{T}_{\sigma_{0|t}} = \mathcal{O}_{\sigma_{0|t}}$. If (8) has a solution for $x(t)$ then it has a solution for $x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ where $u(t) = u^\star_{0|t}$.

Since the optimal control problem (8) explicitly requires that the input $u(t) = u^\star_{0|t}$ and state $x(t+1) = x_{1|t}$ satisfy constraints (8c), Theorem 3 means that the model predictive controller (9) guarantees constraint satisfaction for any state initial state $x(t) \in \mathcal{D}_i(t-t_s)$ in the domain $\mathcal{D}_i(t-t_s)$ of the controller. Corollary 3 below characterizes the domain $\mathcal{D}_i(t–t_s)$ of the model predictive controller when the terminal set $\mathcal{T}_{\sigma_{0|t}}$ is the maximal control switch-invariant set $\mathcal{C}^\infty_{\sigma_{0|t}}$.

Corollary 3: Let $\mathcal{T}_{\sigma_{0|t}} = C^\infty_{\sigma_{0|t}}$ and let $t = t_s$. Then the optimal control problem (8) is defined for all initial states $x(t_s) \in \mathrm{Pre}^d_{\sigma_{0|t}}(C^\infty_{\sigma_{0|t}})$ that can reach $C^\infty_{\sigma_{0|t}}$ in d discrete-time steps i.e. $\mathcal{D}_i(0) = \mathrm{Pre}^d_{\sigma_{0|t}}(C^\infty_{\sigma_{0|t}})$.

Corollary 3 states that if the MPC (9) is engaged immediately after a mode switch $t = t_s$, then its domain is the set $\mathcal{D}_i(0) = \mathrm{Pre}^d_i(C^\infty_i)$. According to Theorem 2, it is only possible to guarantee constraint satisfaction if the initial state $x(t)$ lies inside this set $\mathcal{X}^0_i = \mathrm{Pre}^d_i(C^\infty_i)$. Thus, Corollary 3 implies that the MPC (9) guarantees constraint satisfaction for the system (1) whenever it is possible. Thus, in terms of constraint satisfaction, the MPC (9) is not conservative.

V. Numerical Example: Vehicle Lane Changing

In this section we apply the theory from this paper to a vehicle lane-changing case study. The lateral vehicle dynamics are modeled in continuous-time by

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix}\delta \tag{10}$$

where the state $x = [y, \dot{y}, \psi, \dot{\psi}]^\intercal$ includes the lateral position y, lateral velocity $\dot{y}$, yaw angle ψ, and yaw rate $\dot{\psi}$, and the steering angle $u = \delta$ is the control input. The elements of A and B depend on the vehicle's velocity. Details can be found in [16]. The continuous-time dynamics are discretized with a sample period of 0.2 seconds. The vehicle dynamics do not depend on the mode, but the constraints are mode dependent.

The vehicle has four modes $\mathbb{I} = \{1, 2, 3, 4\}$. Modes $1 \in \mathbb{I}$ and $2 \in \mathbb{I}$ are lane keeping modes for the first and second lanes respectively. Modes $4 \in \mathbb{I}$ and $3 \in \mathbb{I}$ are transition modes, respectively for changing from the first to second lane and for the reverse. The state constraints for mode $1 \in \mathbb{I}$ keep the lateral position y of the vehicle inside the first lane $\mathcal{X}_1 = \{x : -2 \leq y \leq -1\}$. Likewise the state constraints for mode $2 \in \mathbb{I}$ keep the lateral position y of the vehicle inside the second lane $\mathcal{X}_2 = -\mathcal{X}_1$. The state constraints for modes $3, 4 \in \mathbb{I}$ cover both the first and second lane $\mathcal{X}_3 = \mathcal{X}_4 = \{x : -2 \leq y \leq 2\}$. Each set in $\{\mathcal{X}_i\}_{i=,1,2,3,4}$ contains an equilibrium of the vehicle dynamics (10) since the equilibria are of the form $x = [y, 0, 0, 0]^\intercal$.

The transition modes $3, 4 \in \mathbb{I}$ are necessary since the lane sets are disjoint $\mathcal{X}_1 \cap \mathcal{X}_2 = \varnothing$. Thus, it is not possible to directly switch from one lane keeping mode to another. The transition modes $3, 4 \in \mathbb{I}$ relax the lane keeping constraints and allow the vehicle to move between lanes.

The admissible mode switches are given by the graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ shown in Fig. 1. The first lane keeping mode can only switch to the first-to-second transition mode, which can only switch to the second lane keeping mode, implying that $(1, 4), (4, 2) \in \mathbb{E}$. Likewise $(2, 3), (3, 1) \in \mathbb{E}$.

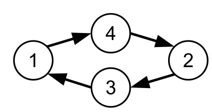

Fig. 1. Graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ of admissible switches between modes. Edges $(i, j) \in \mathbb{E}$ represent when mode switches $i \rightarrow j$ is allowed.

The dwell-time for this example is 2.4 seconds. Fig. 2 shows slices of the control invariant sets $\mathcal{C}_i$ and initial condition sets $\mathcal{X}^0_i$ for each mode $i \in \mathbb{I} = \{1, 2, 3, 4\}$ where the yaw angle and yaw rate are zero $\psi = \dot{\psi} = 0$. The control invariant sets for the first lane keeping mode $1 \in \mathbb{I}$ and the second-to-first lane transition mode $3 \in \mathbb{I}$ are the same $\mathcal{C}_1 = \mathcal{C}_3$. However, the reachable set for the transition mode is larger than that of the lane-keeping mode because the constraints of the transition mode are relaxed relative to those of the lane-keeping mode. For example, $\mathcal{X}_1 \subset \mathcal{X}_3$ implies that $\mathcal{X}^0_1 = \mathrm{Pre}^d(\mathcal{C}_1) \subseteq \mathcal{X}^0_3 = \mathrm{Pre}^d(\mathcal{C}_3)$. In fact, the reachable set $\mathcal{X}^0_1 = \mathrm{Pre}^d(\mathcal{C}_1)$ for the lane keeping mode $1 \in \mathbb{I}$ is the control invariant set $\mathcal{X}^0_1 = \mathcal{C}_1$. This means that the vehicle can only satisfy the lane keeping constraints $\mathcal{X}_1$

if the vehicle state starts inside the control invariant set $\mathcal{C}_1$. On the other hand, if the vehicle starts in the transition mode $3 \in \mathbb{I}$ then the set of initial conditions $\mathcal{X}_3^0 = \mathrm{Pre}^d(\mathcal{C}_3)$ for which it is possible to guarantee constraint satisfaction is larger since the vehicle can use the dwell-time to reach the lane set $\mathcal{X}_1$. A similar relationship holds for the second lane keeping mode $2 \in \mathbb{I}$ and the mode $4 \in \mathbb{I}$ that transitions into the second lane.

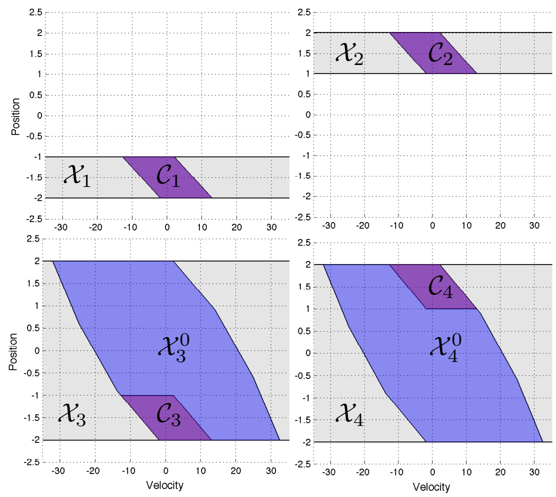

Fig. 2. Sets for each mode $i \in \mathbb{I} = \{1,2,3,4\}$. Grey sets are state constraint sets $\mathcal{X}_i$ i.e. the lane boundaries, red sets are control invariant sets $\mathcal{C}_i$, and blue sets are initial condition sets $\mathcal{X}_i^0 = \mathrm{Pre}^d(\mathcal{C}_i)$.

The control switch-invariant sets $\{\mathcal{C}_i\}_{i=1,2,3,4}$ were used to design a model predictive controller (MPC) for lane changing. The MPC computes the control input by solving (8) where the mode dependent terminal constraint set (8d) is the control switch-invariant set $\mathcal{C}_{\sigma(t)}$ for the current mode $\sigma(t) \in \mathbb{I}$. The horizon of the MPC was $N=12$. The terminal and stage costs are mode dependent and given by

$$p_i(x) = \|x - r_i\|_P^2 \tag{11a}$$

$$q_i(x,u) = \|x - r_i\|_Q^2 + \|u\|_R^2 \tag{11b}$$

where the mode dependent references r_i are $r_1 = r_3 = -1.5$ for the first lane keeping mode $1 \in \mathbb{I}$ and the mode $3 \in \mathbb{I}$ which transitions into the first lane, and $r_2 = r_4 = 1.5$ for the second lane keeping mode $2 \in \mathbb{I}$ and the mode $4 \in \mathbb{I}$ which transitions into the second lane. For this problem, the constraint sets $\mathcal{X}_i$, $\mathcal{U}_i$, and $\mathcal{C}_i$ are mode dependent while the dynamics matrices A and B, and cost matrices Q, R, and P are mode independent.

The penalty matrices Q and R were chosen to provide reference tracking and a smooth transition between lanes. The terminal cost matrix P is the infinite horizon cost matrix for the linear quadratic regulator (LQR) with penalty matrices Q and R. Fig. 3 shows the closed-loop lateral position of the vehicle under the LQR and MPC controllers. While the LQR controller provides smooth tracking of the reference, it does not satisfy the constraints. Similarly if the switch-invariant terminal constraints (8d) were omitted, then the MPC problem (8) could become infeasible after a mode switch. In this case, the MPC, like the LQR, would violate constraints. On the other hand, when the MPC (8) includes the terminal constraints (8d), the state constraints are satisfied, as shown in Fig. 3. This is because the terminal constraints are chosen as control switch-invariant sets $\{\mathcal{C}_i\}_{i=1,2,3,4}$ that satisfy Theorem 3, ensuring persistent feasibility. The theoretical contributions of this paper enabled the design of an MPC that retained the smooth reference tracking of the LQR while modifying the input to ensure that the constraints were always satisfied.

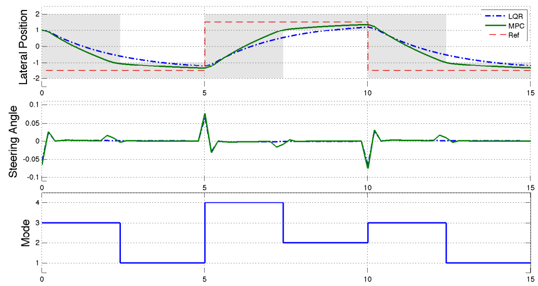

Fig. 3. Lateral position $y(t)$, steering angle $u(t)$, and mode $\sigma(t)$ for a vehicle during a lane change maneuver. Observe that the MPC controller achieves constraint satisfaction for all maneuvers, while the LQR does not.

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