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Wang, Y.; Utsunomiya, K.

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### Abstract

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*Automatisierungstechnik*

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## Article

Yebin Wang\* and Kenji Utsunomiya

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**Abstract:** This work investigates a functional estimation problem for single input single output linear and nonlinear systems. It finds application in enabling acceleration-based semi-active control. Solvability of a linear functional estimation problem is studied from a geometric approach, where the functional dynamics are derived, decomposed, and transformed to expose structural properties. This approach is extended to solve a challenging nonlinear functional observer problem, combining with the exact error linearization. Existence conditions of nonlinear functional observers are established. Simulation verifies existence conditions and demonstrates the effectiveness of the proposed functional observer designs.

**Keywords:** Semi-active, vibration reduction, estimation, nonlinear system, functional observer

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## 1 Introduction

Transportation systems are subject to external disturbances which compromise passenger's ride comfort. A common way to mitigate such a situation is to have a vibration reduction subsystem in place. The use of semi-active actuators, such as Magneto-Rheological (MR) dampers, was originally proposed in [1] to balance the performance of vibration reduction and the system cost. Thanks to the dissipative nature of semi-active actuators, the resultant system enjoys guaranteed stability, and receives longstanding interests from both academia and industry. Nonlinear actuators and inadequate measurements however pose a challenging control problem.

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\*Corresponding author: Yebin Wang, Mitsubishi Electric Research Laboratories, 201 Broadway, Cambridge, MA 02139, USA. email: yebinwang@ieee.org

Kenji Utsunomiya, Advanced Technology R&D Center, Mitsubishi Electric Corporation, 8-1-1, Tsukaguchihonmachi, Amagasaki City, 661-8661, Japan. email: Utsunomiya.Kenji@db.MitsubishiElectric.co.jp

With accelerations and external disturbances, the system is not state observable. This violates the fundamental assumption made in a majority of existing semi-active control designs, e.g. , “clipping” control [1–4], optimal control [5, 6], Lyapunov-based control [7], nonlinear  $\mathcal{H}_\infty$  control [8], Acceleration Driven Damping (ADD) [9], etc. Switching on or off a semi-active actuator according to the sign of its relative velocity, acceleration-based control [10] avoids the pitfall of state unobservability. Its effectiveness is crucially contingent on an accurate estimation of the relative velocity, which is solved as a functional estimation problem.

This work extends discussions on acceleration-based control to a general functional observer design problem. Relevant work includes, but not limited to, functional observer design for linear time-invariant (LTI) systems [11], minimum order functional observer for LTI systems [12], etc. Work [11] establishes necessary and sufficient existence conditions of an  $r$ th order functional observer, where  $r$  is the dimension of functionals. By leveraging additional virtual functionals, work [12] lifts the restriction on the order of the functional observer. Main contributions of this work on the functional estimation are three-fold. First, resorting to a geometric perspective, this work offers a crystal interpretation regarding the solvability of the linear functional estimation problem. Apparent advantages of this approach is that it permits a natural generalization to nonlinear functional observer problem. Different from the linear case, nonlinear functional estimation is more challenging, and its existence conditions are typically sufficient. Second, this work employs the exact error linearization [13, 14], and establishes existence conditions of nonlinear functional observers. Third, simulation is conducted to verify the existence conditions and functional observers.

This paper is organized as follows. Section 2 introduces the semi-active vibration reduction system, and illustrates how the functional estimation enables the acceleration-based control. Section 3 investigates the functional estimation for single input single output (SISO) linear/nonlinear systems. Section 4 presents simulation results. Section 5 concludes this paper.

## 2 Preliminary

This section introduces the semi-active vibration reduction system, and formulates the control problem. Acceleration-based control and the related function observer design are presented.

### 2.1 System Architecture and Modeling

For simplicity, consider a quarter car model which is simplified as a two degree of freedom (2DOF) system as shown in Figure 1. The 2DOF system consists of a first mass ( $m_1$ ), a second mass ( $m_2$ ), a road profile, a controller ( $C$ ), a sensor ( $S$ ), two passive dampers with viscous damping coefficients ( $b_1, b_2$ ), one semi-active damper with adjustable viscous damping coefficient ( $u$ ), and springs ( $k_1, k_2$ ). The system dynamics are given by

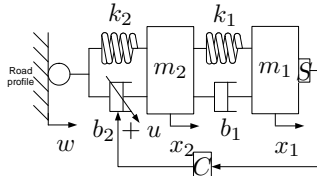


Fig. 1: A 2DOF quarter car model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2(x, \dot{w})u, \\ y &= \dot{x}_2 = h(x, u, w, \dot{w}), \end{aligned} \quad (1)$$

where  $x = [x_1, x_2, \dot{x}_1, \dot{x}_2]^\top = [x_1, x_2, x_3, x_4]^\top$ ,  $w$  is the displacement disturbance from the road profile, and

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1}{m_2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix}, \quad B_2(x, \dot{w}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{x_4 - \dot{w}}{m_2} \end{bmatrix}.$$

The system setup is non-unique. For instance, a semi-active actuator can be placed between  $m_1$  and  $m_2$ . The aforementioned setup is chosen because for certain systems, its dominant resonant mode is result from  $m_2$  and  $k_2$ , and an actuator between  $m_2$  and the road profile is more effective in suppressing the vibration.

System (1) is not uniformly state observable. This fact can be established by verifying that system (1) with

$w = 0$  and  $u = 0$  is not state observable. Given that system state can not be reconstructed from measurements, neither the disturbance  $w$  can be estimated. Readers are referred to [15, 16] for uniform state observability and input observability.

### 2.2 The Control Problem

Vibration reduction is typically formulated as a disturbance attenuation problem as illustrated by Figure 2, where  $G$  is the plant, and  $z$  is the controlled variable, respectively. This work considers the vibration reduction

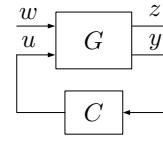


Fig. 2: Disturbance attenuation problem setup [17]

problem roughly stated as follows.

**Problem 2.1.** Given the system (1) and a bounded control set  $U : [b_{min}, b_{max}]$  with  $b_{min}$  and  $b_{max}$  representing the lower and upper bounds of the control, find control  $u \in U$  which minimizes the cost function  $J$

$$J(u) = \int_0^{t_f} |\ddot{x}_1(t)|^2 dt, \quad (2)$$

where  $t_f$  is constant.  $\square$

The cost function (2) does not penalize control efforts. This is not unusual, for example, the time optimal control problem uses the final time as the cost function without penalizing control. As shown later, Problem 2.1 admits a bang-bang control as the time optimal case.

### 2.3 Acceleration-Based Control

A natural treatment of Problem 2.1 is optimal control theory. Optimal open-loop or state feedback control requires to solve an optimization problem recursively [18] or the Hamilton-Jacobi-Bellman equations [19, 20], respectively, neither of which is trivial and realistic. An acceleration-based control has been proposed in [10], and demonstrates good performance, mimicking the structure of the optimal control. Its implementation entails the reconstruction of a relative velocity – a functional estimation problem. The

acceleration-based control and the related functional estimation are recalled for completeness and motivation.

### 2.3.1 Control Structure

Provided that  $w(t)$  is known, Problem 2.1 can be formulated as a constrained optimal control problem. Defining the Hamiltonian

$$H(x, \lambda, u, w, \dot{w}) = |\ddot{x}_1(t)|^2 + \lambda^\top (Ax + B_1 w + B_2(x, \dot{w})u),$$

we have the structure of the optimal control which minimizes  $H(x, \lambda, u, t)$

$$u^* = \begin{cases} b_{max}, & \text{if } \lambda^\top B_2(x, \dot{w}) \leq 0, \\ b_{min}, & \text{if } \lambda^\top B_2(x, \dot{w}) \geq 0. \end{cases}$$

Given the costate  $\lambda = [\lambda_1, \dots, \lambda_4]^\top$ , the optimal control can be written as

$$u^* = \begin{cases} b_{max}, & \text{if } \lambda_4(x_4 - \dot{w}) \geq 0, \\ b_{min}, & \text{if } \lambda_4(x_4 - \dot{w}) \leq 0. \end{cases} \quad (3)$$

Because system (1) is not state observable, the optimal control (3) is not realizable. Instead, [10] approximates (3) as the following acceleration-based control

$$u = \begin{cases} b_{max}, & \text{if } (c_1 \hat{x}_1 + c_2 \dot{x}_4 + \hat{x}_4)(\hat{x}_4 - \dot{w}) \geq 0, \\ b_{min}, & \text{otherwise,} \end{cases} \quad (4)$$

where  $c_1, c_2$  are constants to be tuned,  $\hat{x}_1$  is the estimated acceleration of the first mass,  $\hat{x}_4$  is the estimated velocity of the second mass, and  $\hat{x}_4 - \dot{w}$  is the estimated relative velocity of the semi-active actuator. Readers are referred to [10] for the essence of (4).

### 2.3.2 Estimation of the Relative Velocity

We rearrange the dynamics of  $x_4$

$$m_2 \dot{x}_4 = -m_1 \ddot{x}_1 - k_2(x_2 - w) - u(t)(\dot{x}_2 - \dot{w}),$$

and derive the dynamics of the relative position  $\eta = x_2 - w$  as follows

$$\dot{\eta} = \frac{-1}{u(t)}(k_2 \eta + m_1 \ddot{x}_1 + m_2 y). \quad (5)$$

Given the transfer function from the second mass displacement to the first mass displacement

$$G_2 = \frac{X_1(s)}{X_2(s)} = \frac{b_1 s + k_1}{m_1 s^2 + b_1 s + k_1}.$$

we design a linear time-varying filter

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_3, \\ \dot{\hat{x}}_3 &= -\frac{k_1}{m_1} \hat{x}_1 - \frac{b_1}{m_1} \hat{x}_3 + \frac{1}{m_1} y, \\ \dot{\hat{\eta}} &= \frac{-1}{u(t)}(k_2 \hat{\eta} + m_1(k_1 \hat{x}_1 + b_1 \hat{x}_3) + m_2 y), \end{aligned} \quad (6)$$

where the output  $\hat{\eta}$  is globally exponentially convergent to  $\eta$ . Since  $\hat{x}_1, \hat{x}_3, \hat{\eta}$  exponentially converge to  $x_1, x_3, \eta$ , respectively,  $|\dot{\eta}(t) - \hat{\eta}(t)| \rightarrow 0$  as  $t \rightarrow \infty$ . The filter is open-loop, and the convergent rate of the estimation error cannot be designed.

## 3 Functional Estimation

The aforementioned estimation method can be generalized to solve a class of problems: functional estimation for unobservable systems. This section proposes a functional estimation method, based on deriving and transforming the dynamics of the functional. Without loss of generality, this work considers the functional estimation of SISO systems. The idea can be extended to systems having multi-input and multi-output, or unknown inputs.

### 3.1 Linear System Case

Consider an SISO linear time-invariant (LTI) system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ z &= Lx, \end{aligned} \quad (7)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}$  is the input,  $y \in \mathbb{R}$  is the output,  $z \in \mathbb{R}$  is the functional to be estimated, and  $A, B, C, L$  are matrices with appropriate dimensions.

**Definition 3.1.** *Given an SISO LTI system (7), the  $z$ -dynamics takes the following state space form*

$$\begin{aligned} \dot{\xi} &= Q\xi + Mu, \\ z &= \xi_1, \end{aligned} \quad (8)$$

where  $\xi = [\xi_1, \dots, \xi_r]^\top$  and  $Q \in \mathbb{R}^{r \times r}, M \in \mathbb{R}^r$  with its  $i$ th element denoted by  $M_i$ .

System (7) is a trivial representation of the  $z$ -dynamics. The  $z$ -dynamics are generally non-unique. The following result states how to derive the  $z$ -dynamics with the minimal order  $r$ , abbreviated to the minimal  $z$ -dynamics.

**Proposition 3.2.** *Given system (7), its minimal  $z$ -dynamics are represented by the  $r$ th order  $\xi$ -system (8) if and only if*

- (i)  $\dim(\text{span}\{L, LA, \dots, LA^{r-1}\}) = r$ ;
- (ii)  $LA^r \in \text{span}\{L, LA, \dots, LA^{r-1}\}$ .

*Proof.* Sufficiency: With Condition (i), one can define state variables  $\xi$  as follows

$$\xi = [Lx, LAx, \dots, LA^{r-1}x]^\top.$$

Taking the time derivative of  $\xi$  gives

$$\begin{aligned} \dot{\xi}_1 &= LAx + LBu = \xi_2 + M_1u, \\ &\vdots \\ \dot{\xi}_r &= LA^r x + LA^{r-1}Bu = LA^r x + M_r u. \end{aligned}$$

With Condition (ii), we have  $LA^r = \sum_{k=0}^{r-1} a_k LA^k$ , and

$$\dot{\xi}_r = \sum_{k=0}^{r-1} a_k \xi_{k+1} + M_r u.$$

Condition (i) implies that for any  $1 \leq k \leq r-1$ ,  $LA^k \notin \text{span}\{L, \dots, LA^{k-1}\}$ . Hence, for any subsystem  $\xi_{[1,k]} = [\xi_1, \dots, \xi_k]^\top$  with  $1 \leq k \leq r-1$ , its state space representation is given by

$$\dot{\xi}_{[1,k]} = A_{[1,k]}\xi_{[1,k]} + B_{[1,k]}\xi_{k+1} + M_{[1,k]}u,$$

where  $M_{[1,k]} = [M_1, \dots, M_k]^\top$ , and

$$A_{[1,k]} = \begin{bmatrix} 0 & \mathbf{I}_{k-1} \\ 0 & \mathbf{0}_{k-1} \end{bmatrix} \xi_{[1,k]}, \quad B_{[1,k]} = \begin{bmatrix} \mathbf{0}_{k-1} \\ 1 \end{bmatrix}.$$

Clearly,  $\xi_{[1,k]}$ -system is not in the form of (8), and we conclude  $r$  is the minimal order.

Necessity: Similar to the proof of Sufficiency, one can verify that: if Condition (i) does not hold, then  $\xi$ -system (8) is not the minimal  $z$ -dynamics; and if Condition (ii) does not hold,  $\xi$ -system is not in the form (8), and does not represent  $z$ -dynamics.  $\square$

**Remark 3.3.** *Proof of Proposition 3.2 indicates that the minimal  $z$ -dynamics exactly represent the observable subsystem of the following system*

$$\dot{x} = Ax + Bu, \quad y = Lx.$$

The observable space is denoted by

$$\mathcal{O}_z = \text{span}\{L, \dots, LA^{r-1}\}.$$

**Remark 3.4.** *The  $z$ -dynamics can be alternatively defined as follows*

$$\begin{aligned} \dot{\xi} &= Q\xi + Mu + Hy, \\ z &= \xi_1, \end{aligned} \quad (9)$$

where  $H \in \mathbb{R}^r$ . The existence conditions can be similarly derived by following the proof of Proposition 3.2.

If  $Q$  is Hurwitz, there exists an open-loop estimator

$$\begin{aligned} \dot{\hat{\xi}} &= Q\hat{\xi} + Mu, \\ \hat{z} &= \hat{\xi}_1. \end{aligned} \quad (10)$$

The zero solution of the resultant estimation error dynamics converges exponentially. The convergence rate is dominated by the maximum eigenvalue of  $Q$ , e.g.  $\lambda_{\max}(Q)$ , which is a key limitation. Next, we treat the case when  $Q$  is not Hurwitz, and establish existence conditions of a stable functional estimator.

Following notations are introduced to make subsequent discussions succinct. For system (7) with output  $y$ , its observable subspace is

$$\mathcal{O} = \text{span}\{C, \dots, CA^{r_0-1}\},$$

and its  $r_2$ -dimension observable subspace is

$$\mathcal{O}_{r_2} = \text{span}\{C, \dots, CA^{r_2-1}\}.$$

A functional observer form is defined as follows

$$\begin{aligned} \dot{\bar{\xi}}^1 &= Q^{11}\bar{\xi}^1 + Q^{12}\bar{\xi}^2 + M^1u, \\ \dot{\bar{\xi}}^2 &= Q^{22}\bar{\xi}^2 + M^2u, \\ z &= C_z\bar{\xi}^1, \\ y &= C_y\bar{\xi}^2, \end{aligned} \quad (11)$$

where  $\bar{\xi}^1 \in \mathbb{R}^{r_1}$ ,  $\bar{\xi}^2 \in \mathbb{R}^{r_2}$ ,  $Q^{11}$  is Hurwitz,  $M^1 = [M_1^1, \dots, M_{r_1}^1]^\top$ ,  $M^2 = [M_1^2, \dots, M_{r_2}^2]^\top$ ,  $C_z = [1, \mathbf{0}_{r_1-1}]$ ,  $C_y = [1, \mathbf{0}_{r_2-1}]$ , and  $(C_y, Q^{22})$  is observable. Given (11), a functional observer is designed as follows

$$\begin{aligned} \dot{\hat{\xi}}^1 &= Q^{11}\hat{\xi}^1 + Q^{12}\hat{\xi}^2 + M^1u, \\ \dot{\hat{\xi}}^2 &= Q^{22}\hat{\xi}^2 + M^2u + G(y - \hat{y}), \\ \hat{z} &= C_z\hat{\xi}^1, \\ \hat{y} &= C_y\hat{\xi}^2, \end{aligned} \quad (12)$$

where  $G$  makes  $Q_c^{22} = Q^{22} - GC_y$  Hurwitz. The resultant estimation error dynamics are

$$\begin{aligned} \dot{\tilde{\xi}}^1 &= Q^{11}\tilde{\xi}^1 + Q^{12}\tilde{\xi}^2, \\ \dot{\tilde{\xi}}^2 &= (Q^{22} - GC_y)\tilde{\xi}^2, \end{aligned} \quad (13)$$

where  $\tilde{\xi}^1 = \bar{\xi}^1 - \hat{\xi}^1$ ,  $\tilde{\xi}^2 = \bar{\xi}^2 - \hat{\xi}^2$ . The zero solution of (13) is globally exponentially stable (GES), which implies  $\hat{z}(t) \rightarrow z(t)$  as  $t \rightarrow \infty$ .

We have the following result regarding whether the functional can be estimated.

**Proposition 3.5.** *Given system (7) and the minimal  $z$ -dynamics (8), the functional  $z$  can be estimated if and only if (8) is a subsystem of the system (11) with  $r_2 = r_o$ , and the state is given by*

$$\begin{aligned}\bar{\xi}^1 &= [\bar{\xi}_1^1, \dots, \bar{\xi}_{r_1}^1]^\top = [Lx, \dots, LA^{r_1-1}x]^\top, \\ \bar{\xi}^2 &= [\bar{\xi}_1^2, \dots, \bar{\xi}_{r_o}^2]^\top = [Cx, \dots, CA^{r_o-1}x]^\top.\end{aligned}\quad (14)$$

*Proof.* Sufficiency: Assume that  $z$ -dynamics are part of system (11), i.e.,  $\xi = T\bar{\xi}$  with  $T$  being a linear map. The functional observer (12) yields convergent estimates  $\hat{\bar{\xi}}$  of  $\bar{\xi}$ , which implies  $\hat{\xi}(t) \rightarrow \xi(t)$  as  $t \rightarrow \infty$ .

Necessity: assume the existence of a functional observer to estimate  $z$ . Given system (7), one can always define a set of coordinates of its observable subspace as

$$\bar{\xi}^2 = [\bar{\xi}_1^2, \dots, \bar{\xi}_{r_o}^2]^\top = [Cx, \dots, CA^{r_o-1}x]^\top.$$

On the other hand, Proposition 3.2 shows that coordinates of the  $z$ -dynamics (8) are given by

$$\xi = [Lx, \dots, LA^{r-1}x].$$

It is always possible to pick  $r_1 \leq r$  states out of  $\xi$  to form  $\bar{\xi}^1$ , which are linearly independent of  $\bar{\xi}^2$ . For system (7), one can define a change of coordinates as follow

$$\bar{\xi} = [(\bar{\xi}^1)^\top, (\bar{\xi}^2)^\top, (\bar{\xi}^3)^\top]^\top = T_n x \in \mathbb{R}^n,$$

where  $\bar{\xi}^3 \in (\mathcal{O}_z \cup \mathcal{O})^\perp$ . That is:  $\bar{\xi}^3$  influences neither  $z$ -dynamics nor the observable subsystem  $\bar{\xi}^2$ .

We ought to derive the expression of system (7) in the new coordinates  $\bar{\xi}$ . It is clear that the  $\bar{\xi}^2$ -dynamic should have the structure as specified in (11). The state  $\bar{\xi}^1$  evolves in the subspace  $\mathcal{O}_z \cup \mathcal{O}$ , which is not impacted by  $\bar{\xi}^3$ . Hence,  $\bar{\xi}^1$ -dynamics takes the form as (11). State  $\bar{\xi}^3$  evolves in the subspace  $(\mathcal{O}_z \cup \mathcal{O})^\perp$ , and thus its dynamics has to be written as  $\dot{\bar{\xi}}^3 = Q^{33}\bar{\xi}^3 + M^3u$ . Meanwhile, system (11) includes  $z$ -dynamics as a subsystem.

Next we show that  $[Lx, \dots, LA^{r_1-1}x]^\top$  has to be one representation of  $\bar{\xi}^1$ . The state space corresponding to the  $z$ -dynamics, or the observable space  $\mathcal{O}_z$ , can be decomposed into two subspaces:  $\mathcal{S}_o = \mathcal{O}_z \cap \mathcal{O}$ , and  $\mathcal{S}_u = \mathcal{O}_z \setminus \mathcal{S}_o$ . It is clear that  $\dim \mathcal{S}_u = r_1$ . We only need to show that the coordinates of  $\mathcal{S}_u$  are given in (14). One can readily verify that if  $LA^k \in \text{span} \{L, \dots, LA^{k-1}\} \cup \mathcal{O}$ , so does  $LA^{k+1}$  for  $k \leq r-2$ . That is, if  $\dim(\mathcal{O}_z \cup \mathcal{O}) = r_1 + r_o$ , then

$$\text{span} \{L, \dots, LA^{r-1}\} \cup \mathcal{O} = \text{span} \{L, \dots, LA^{r_1-1}\} \cup \mathcal{O}.$$

Hence,  $[Lx, \dots, LA^{r_1-1}x]^\top$  has to be one representation of  $\bar{\xi}^1$ . This completes the proof.  $\square$

**Remark 3.6.** *If  $\mathcal{O}_z \cap \mathcal{O} = \emptyset$ ,  $z$ -dynamics are completely unobservable. The functional  $z$  can be reconstructed, at best, by the open-loop estimator (10).*

Proposition 3.5 establishes necessary and sufficient conditions for functional estimation. The order of the resultant estimator is likely higher than the  $z$ -dynamics.

**Example 3.7.** *Consider the following LTI system*

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= [1 \ 0 \ 0 \ 0 \ 0] x, \\ z &= [0 \ 1 \ 0 \ 1 \ 0] x.\end{aligned}\quad (15)$$

*Verification of Proposition 3.2 confirms  $r = 4$  and*

$$\begin{aligned}L &= [0 \ 1 \ 0 \ 1 \ 0], \\ LA &= [0 \ 0 \ 1 \ 0 \ 1], \\ LA^2 &= [0 \ 0 \ 0 \ -1 \ -2], \\ LA^3 &= [0 \ 0 \ 0 \ 2 \ 3], \\ LA^4 &= [0 \ 0 \ 0 \ -3 \ -4],\end{aligned}$$

*where  $LA^4 = -LA^2 - 2LA^3 \in \text{span} \{L, LA, LA^2, LA^3\}$ . We therefore have the minimal  $z$ -dynamics*

$$\begin{aligned}\dot{\bar{\xi}} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}}_Q \bar{\xi} + Mu, \\ z &= \xi_1,\end{aligned}$$

*where  $Q$  is not Hurwitz. The functional  $z$  can not be reconstructed by the open-loop estimator (10).*

*Proposition 3.5 holds with state variables*

$$\bar{\xi}^1 = [Lx, LAx]^\top, \quad \bar{\xi}^2 = [Cx, CAx, CA^2x]^\top,$$

*and the dynamics are given by*

$$\begin{aligned}\dot{\bar{\xi}}^1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \bar{\xi}^1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \bar{\xi}^2 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \\ \dot{\bar{\xi}}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \bar{\xi}^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.\end{aligned}$$

The functional can be reconstructed by the estimator

$$\begin{aligned}\dot{\hat{\xi}}^1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \hat{\xi}^1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \hat{\xi}^2 + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \\ \dot{\hat{\xi}}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{\xi}^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + G(y - \hat{y}), \\ \hat{y} &= \hat{\xi}_1^2, \\ \hat{z} &= \hat{\xi}_1^1,\end{aligned}$$

where  $G$  stabilizes the zero solution of  $\tilde{\xi}^2$ -dynamics.  $\square$

### 3.2 Nonlinear System Case

We first introduce a few notations. Given a  $C^\infty$  vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and a  $C^\infty$  function  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ , the function  $L_f \alpha = \frac{\partial \alpha}{\partial x} f$  is the *Lie derivative* of  $\alpha$  along  $f$ . Repeated Lie derivatives are defined as

$$L_f^k \alpha = L_f(L_f^{k-1} \alpha), \quad k \geq 1$$

with  $L_f^0 \alpha = \alpha$ .

Consider the following nonlinear system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u, \\ y &= c(x), \\ z &= l(x),\end{aligned}\tag{16}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ ,  $y \in \mathbb{R}$ ,  $f(x), g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are smooth vector fields,  $c(x), l(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  are smooth scalar functions, and  $z \in \mathbb{R}$  is the functional. We define  $z$ -dynamics for system (16).

**Definition 3.8.** *Given system (16), the  $z$ -dynamics take the following state space form*

$$\begin{aligned}\dot{\xi} &= Q(\xi) + M(\xi)u, \\ z &= \xi_1,\end{aligned}\tag{17}$$

where  $\xi = [\xi_1, \dots, \xi_r]^\top$  and  $Q, M$  are smooth vector fields in the form of

$$Q(\xi) = \begin{bmatrix} \xi_2 \\ \vdots \\ \varphi(\xi) \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} M_1(\xi) \\ \vdots \\ M_r(\xi) \end{bmatrix}.$$

Let  $\mathcal{O}_z = \text{span} \{dl(x), \dots, dL_f^{r-1}l(x)\}$  be the observable space of system (16) with output  $z = l(x)$ . The following result establishes existence conditions of  $z$ -dynamics (17).

**Proposition 3.9.** *Given system (16), its  $z$ -dynamics are represented by an  $r$ -th order system (17) if and only if*

- (i)  $\dim(\mathcal{O}_z) = r$ ;
- (ii)  $dL_g L_f^k l(x) \in \mathcal{O}_z$  for  $0 \leq k \leq r-1$ ;
- (iii)  $dL_f^r l(x) \in \mathcal{O}_z$ .

*Proof.* Sufficiency: Condition (i) ensures that

$$\xi(x) = [l(x), \dots, L_f^{r-1}l(x)]^\top$$

defines coordinates of the space  $\mathcal{O}_z$ . We have

$$\begin{aligned}\xi_1 &= \xi_2 + L_g l(x)u, \\ &\vdots \\ \dot{\xi}_r &= L_f^r l(x) + L_g L_f^{r-1} l(x)u.\end{aligned}$$

Conditions (ii) and (iii) assure that  $L_g L_f^k l(x)$  and  $L_f^r l(x)$  is a function of  $\xi$ , respectively. Sufficiency is shown.

Necessity: Assume that the  $z$ -dynamics are given by (17). One can verify Conditions (i)-(iii).  $\square$

**Remark 3.10.** *The  $z$ -dynamics (17) can be generalized to the following form*

$$\begin{aligned}\dot{\xi} &= Q(\xi, y) + M(\xi, y)u, \\ z &= \xi_1,\end{aligned}\tag{18}$$

where  $Q(\xi, y) = [\xi_2, \dots, \varphi(\xi, y)]^\top$ ,  $M(\xi, y) = [M_1(\xi, y), \dots, M_r(\xi, y)]^\top$ . Similar to Proposition 3.9, existence conditions of the  $z$ -dynamics in the form of (18) can be derived.

**Remark 3.11.** *Functional estimation for nonlinear systems is significantly challenging, compared with the linear case. To be consistent with the linear case where the  $Q$  matrix in (8) is Hurwitz, we assume that the zero solution of the uncontrolled  $z$ -dynamics  $\dot{\xi} = Q(\xi)$  is GES. Given an estimator*

$$\dot{\hat{\xi}} = Q(\hat{\xi}) + M(\hat{\xi})u, \quad \hat{z} = \hat{\xi}_1,$$

the stability analysis of the estimation error  $\xi - \hat{\xi}$  is hardly conclusive.

The  $z$ -dynamics (17) do not necessarily permit the functional estimation. It is however always possible if the  $z$ -dynamics are diffeomorphic to an observer form

$$\begin{aligned}\dot{\xi} &= Q\xi + Mu, \\ z &= \xi_1,\end{aligned}\tag{19}$$

where  $Q$  and  $M$  are constant matrices with  $Q$  Hurwitz. Recall the well-established exact error linearization (EEL) to perform observer design for nonlinear systems, e.g. [13]. Applying the EEL to the  $z$ -dynamics with  $z$  being treated



as the virtual output, existence conditions of the observer form (19) can be readily established. The observer form (19) guarantees the existence of a stable open-loop estimator for estimation of  $z$ .

**Proposition 3.12.** *System (16) induces the  $z$ -dynamics in the observer form (19), if and only if*

- (i)  $\dim(\mathcal{O}_z) = r$ ;
- (ii)  $L_f^r l(x) = \sum_{i=0}^{r-1} \alpha_i L_f^i l(x)$  with the characteristics equation  $\sum_{i=0}^{r-1} \alpha_i s^i = 0$  being Hurwitz;
- (iii) for  $0 \leq k \leq r-1$ ,  $dL_g L_f^k l(x) = 0$ .

*Proof.* Sufficiency: Condition (i) suggests that  $\xi = [l(x), \dots, L_f^{r-1} l(x)]^\top$  are coordinates of the observable subsystem associated with the virtual output  $z$ . Conditions (ii) ascertains that  $Q$  is Hurwitz in the  $\xi$ -coordinates; and Condition (iii) guarantees that  $M$  is constant.

Necessity: Given system (16) and its subsystem in the observer form (19), one can always choose  $(n-r)$  independent state variables  $\bar{\xi}(x)$  from the original state  $x$ , such that  $[\xi^\top, \bar{\xi}^\top(x)]^\top$  define new coordinates of system (16). Under the new coordinates, system (16) is written as follows

$$\begin{aligned}\dot{\xi} &= Q\xi + Mu, \\ \dot{\bar{\xi}} &= \bar{f}(\xi, \bar{\xi}) + \bar{g}(\xi, \bar{\xi})u, \\ y &= c(\xi, \bar{\xi}), \\ z &= \xi_1.\end{aligned}$$

Vector fields  $f, g$  in the new coordinates are given by  $f(\xi, \bar{\xi}) = [(Q\xi)^\top, \bar{f}(\xi, \bar{\xi})^\top]^\top$ ,  $g(\xi, \bar{\xi}) = [M^\top, \bar{g}(\xi, \bar{\xi})^\top]^\top$ . With  $f(\xi, \bar{\xi}), g(\xi, \bar{\xi})$ , Conditions (i)-(iii) are verified. Necessity is shown.  $\square$

Proposition 3.12 is restrictive, not only because it requires the  $z$ -dynamics being exactly linearized, but also because it needs the linearized  $z$ -dynamics are stable to allow functional estimation. Like Proposition 3.5, the restriction can be lifted by decomposing the  $z$ -dynamics into two parts: observable and unobservable through  $y$ .

Generalize the observer form (19) to the partial observer form:

$$\begin{aligned}\dot{\xi}^1 &= Q^{11}\xi^1 + Q^{12}(\xi^2) + M^1 u, \\ \dot{\xi}^2 &= Q^{22}(\xi^2) + M^2(\xi^2)u, \\ y &= \xi_1^2, \\ z &= \xi_1^1,\end{aligned}\tag{20}$$

where  $Q^{11}$  is Hurwitz,  $M^1$  is constant, and

$$\begin{aligned}\xi^1 &= [l(x), \dots, L_f^{r_1-1} l(x)]^\top, \\ \xi^2 &= [c(x), \dots, L_f^{r_2-1} c(x)]^\top, \\ Q^{12}(\xi^2) &= [0, \dots, Q_{r_1}^{12}(\xi^2)]^\top, \\ Q^{22}(\xi^2) &= \begin{bmatrix} \xi_2^2 \\ \vdots \\ \varphi(\xi^2) \end{bmatrix}, \quad M^2(\xi^2) = \begin{bmatrix} M_{r_1}^2(\xi_1^2) \\ \vdots \\ M_{r_2}^2(\xi^2) \end{bmatrix},\end{aligned}$$

with  $r_1 + r_2 = r$ .

**Proposition 3.13.** *System (16) induces the  $z$ -dynamics in the form (20) for a pair  $(r_1, r_2)$  if and only if,*

- (i)  $\dim(\mathcal{O}_{z, r_1}) + \dim(\mathcal{O}_{r_2}) = r$  where

$$\begin{aligned}\mathcal{O}_{z, r_1} &= \text{span} \{dl(x), \dots, dL_f^{r_1-1} l(x)\}, \\ \mathcal{O}_{r_2} &= \text{span} \{dc(x), \dots, dL_f^{r_2-1} c(x)\};\end{aligned}$$

- (ii) *There exists  $\alpha_i$  for  $0 \leq i \leq r-1$  such that the characteristics equation  $\sum_{i=0}^{r-1} \alpha_i s^i = 0$  is Hurwitz; and*

$$\begin{aligned}\sum_{i=0}^{r_1-1} \alpha_i dL_f^i l(x) &= dL_f^{r_1} l(x) \pmod{\{dc(x), \dots, dL_f^{r_2-1} c(x)\}};\end{aligned}$$

- (iii) for  $0 \leq k \leq r_1-1$ ,  $dL_g L_f^k l(x) = 0$ ;

- (iv) for  $0 \leq k \leq r_1-1$ ,

$$dL_g L_f^k c(x) \in \text{span} \{dc(x), \dots, dL_f^k c(x)\};$$

- (v)  $dL_f^{r_2} c(x) \in \mathcal{O}_{r_2}$ .

*Proof.* Sufficiency: From Conditions (i), one known  $\xi^1$  and  $\xi^2$  are independent variables. For  $\xi^1$ -subsystem, using Conditions (ii)-(iii), we have

$$\begin{aligned}\dot{\xi}_1^1 &= L_f l(x) + L_g l(x)u = \xi_2 + M_1^1 u, \\ &\vdots \\ \dot{\xi}_{r_1}^1 &= L_f^{r_1} l(x) + L_g L_f^{r_1-1} l(x)u \\ &= \sum_{i=0}^{r_1-1} \alpha_i dL_f^i l(x) + Q_{r_1}^{12}(\xi^2) + M_{r_1}^1 u.\end{aligned}$$

For  $\xi^2$ -subsystem, using Conditions (iv)-(v), we have

$$\begin{aligned}\dot{\xi}_1^2 &= L_f c(x) + L_g c(x)u = \xi_2^2 + M_1^2(\xi_1^2), \\ &\vdots \\ \dot{\xi}_{r_2}^2 &= L_f^{r_2} c(x) + L_g L_f^{r_2-1} c(x)u = \varphi(\xi^2) + M_{r_2}^2(\xi^2)u\end{aligned}$$

The system with state variables  $\xi^1$  and  $\xi^2$  admits the partial observer form (20).

Necessity: Given (20),  $\xi^1, \xi^2$  are independent variables. One can construct new coordinates for system (16)

$$\xi = [(\xi^1)^\top, (\xi^2)^\top, (\xi^3)^\top] \in \mathbb{R}^n,$$

where  $\xi^3$  are selected from  $x$  to ensure the non-singularity of  $\partial\xi/\partial x$ . Vector fields  $f, g$ , in the new coordinates, are represented by

$$f(\xi) = \begin{bmatrix} Q^{11}\xi^1 + Q^{12}(\xi^2) \\ Q^{22}(\xi^2) \\ f_3(\xi) \end{bmatrix}, \quad g(\xi) = \begin{bmatrix} M^1 \\ M^2(\xi^2) \\ g_3(\xi) \end{bmatrix},$$

where  $f_3$  and  $g_3$  are vectors of certain functions. With  $f(\xi)$  and  $g(\xi)$ , one can compute  $L_f^k l(\xi)$  for  $0 \leq k \leq r_1$

$$\begin{aligned} L_f l(\xi) &= \xi_2^1, \\ &\vdots \\ L_f^{r_1-1} l(\xi) &= \xi_{r_1}^1, \\ L_f^{r_1-1} l(\xi) &= \sum_{i=0}^{r_1-1} \alpha_i \xi_{i+1}^1 + Q_{r_1}^{12}(\xi^2) \end{aligned}$$

and  $L_f^k c(\xi)$  for  $0 \leq k \leq r_2$

$$\begin{aligned} L_f c(\xi) &= \xi_2^2, \\ &\vdots \\ L_f^{r_2-1} c(\xi) &= \xi_{r_2}^2, \\ L_f^{r_2-1} c(\xi) &= \varphi(\xi^2). \end{aligned}$$

Conditions (i)-(ii) and (v) are verified. Calculating

$$\begin{aligned} L_g L_f^k l(\xi) &= M_{k+1}^1, \quad 0 \leq k \leq r_1 - 1, \\ L_g L_f^k c(\xi) &= M_{k+1}^2(\xi_1^2, \dots, \xi_{k+1}^2), \quad 0 \leq k \leq r_2 - 1, \end{aligned}$$

one verifies Conditions (iii)-(iv). Necessity is proven.  $\square$

Based on (20), we propose the following estimator

$$\begin{aligned} \dot{\hat{\xi}}^1 &= Q^{11}\hat{\xi}^1 + Q^{12}(\hat{\xi}^2) + M^1 u, \\ \dot{\hat{\xi}}^2 &= Q^{22}(\hat{\xi}^2) + M^2(\hat{\xi}^2)u + S^{-1}C_y^\top(y - \hat{y}), \\ \hat{y} &= C_y \hat{\xi}^2, \\ \hat{z} &= \hat{\xi}_1^1, \end{aligned} \quad (21)$$

where  $C_y = [1, 0, \dots, 0] \in \mathbb{R}^{r_2}$  and  $S$  is the solution of

$$\theta S + A_y^\top S + S A_y - C_y^\top C_y = 0,$$

with a sufficiently large  $\theta$  and

$$A_y = \begin{bmatrix} 0 & \mathbf{I}_{r_2-1} \\ 0 & 0 \end{bmatrix}.$$

Under certain assumptions, (21) yields convergent estimation error dynamics. We have the following result.

**Proposition 3.14.** *For system, assume that (20) is globally defined and  $\xi^2$  is uniformly observable through  $y$  for all  $u$ , and  $Q_{r_1}^{12}(\xi^2)$ ,  $\varphi(\xi^2)$ , and  $M^2(\xi^2)$  are globally Lipschitz with respect to their corresponding arguments, i.e.,*

$$\begin{aligned} Q^{12}(\xi^2) - Q^{12}(\hat{\xi}^2) &\leq L_1 \|\tilde{\xi}^2\|, \\ \varphi(\xi^2) - \varphi(\hat{\xi}^2) &\leq L_2 \|\tilde{\xi}^2\|, \\ M^2(\xi^2) - M^2(\hat{\xi}^2) &\leq L_3 \|\tilde{\xi}^2\|, \end{aligned}$$

where  $L_1, L_2, L_3$  are positive constants. Given the functional estimator (21), the zero solution of the resultant estimation error dynamics is GES.

*Proof.* Write the estimation error dynamics as follows

$$\begin{aligned} \dot{\tilde{\xi}}^1 &= Q^{11}\tilde{\xi}^1 + Q^{12}(\xi^2) - Q^{12}(\hat{\xi}^2), \\ \dot{\tilde{\xi}}^2 &= Q^{22}(\xi^2) - Q^{22}(\hat{\xi}^2) + (M^2(\xi^2) - M^2(\hat{\xi}^2))u \\ &\quad + S^{-1}C_y^\top(y - \hat{y}), \\ \dot{\hat{y}} &= C_y \tilde{\xi}^2 \end{aligned} \quad (22)$$

The  $\tilde{\xi}^2$ -dynamics are shown GES in [15]. Since  $Q^{11}$  is Hurwitz,  $\tilde{\xi}^1 = Q^{11}\tilde{\xi}^1$  is GES. Rearrange  $\tilde{\xi}^1$  as

$$\dot{\tilde{\xi}}^1 = Q^{11}\tilde{\xi}^1 + d(t),$$

where  $|d(t)| \leq L \|\tilde{\xi}^2\|$  exponentially decays. We conclude that  $\tilde{\xi}^1(t)$  converges to zero exponentially for any  $\tilde{\xi}^1(0) \in \mathbb{R}^{r_1}$ . Estimation error dynamics (22) are GES.  $\square$

## 4 Simulation

This section presents two examples. The semi-active example demonstrates the effectiveness of the acceleration-based control, where the functional observer estimates the relate velocity. A numerical example verifies existence conditions of partial observer form and the proposed functional observer design.

### 4.1 Semi-Active Example

Extensive simulation and experiments were carried out to validate the proposed design. No experimental details are offered here for the protection of proprietary information. The simulated system (1) has parameter values:  $m_1 = 81.16kg$ ,  $m_2 = 127.54kg$ ,  $k_1 = 7.54e4N/m$ ,  $k_2 = 3.02e4N/m$ ,  $b_1 = 580Ns/m$ ,  $b_2 = 290Ns/m$ ,  $b_{min} = 175Ns/m$ ,  $b_{max} = 2.9e3Ns/m$ . The system is subject to a disturbance shown in Figure 3.

We compare the performance of three cases: Case 1: the passive, i.e., system (1) with  $u = 0$ ; Case 2: system (1)

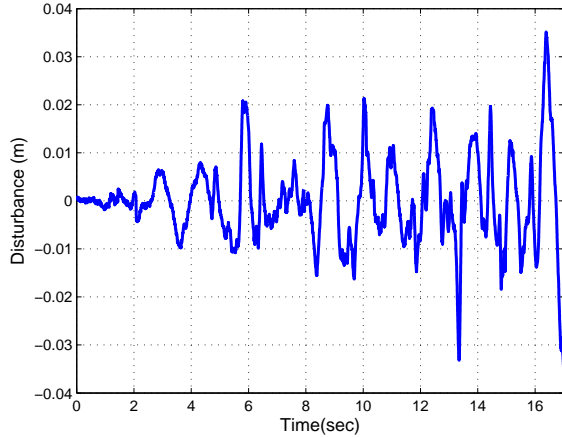


Fig. 3: the disturbance data

with the control (4) and full state; Case 3: the control (4) and the estimator (6). Simulation results, provided in Table 1 and Figure 4, show that Case 3 achieves almost the same performance as Case 2, and leads to 46% vibration reduction over the passive system. In experiment, control (4) and the estimator (6) are implemented in embedded platforms and compared with the passive case. Both experiment and simulation results coincide.

Table 1: Simulation results

Control	Cost $\ \ddot{x}_1\ _2$
Passive	2.6496m/s <sup>2</sup>
Semi-active with (4) and full state	1.4025m/s <sup>2</sup>
Semi-active with (4) and (6)	1.4097m/s <sup>2</sup>

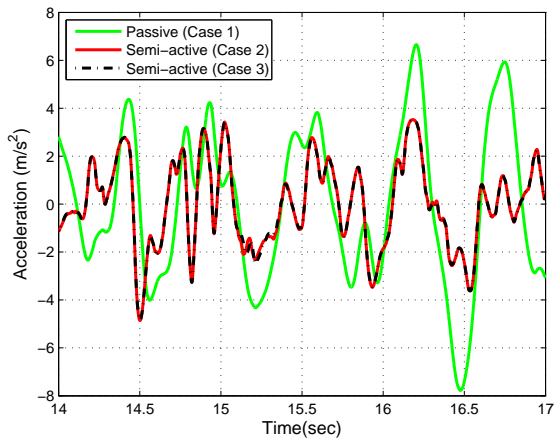


Fig. 4: the performance of the passive and semi-active vibration reduction systems

## 4.2 Numerical Example

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= x_1^1, \\ z &= x_1^2, \end{aligned} \quad (23)$$

where  $x = [(x^1)^\top, (x^2)^\top]^\top$  with  $x^1 \in \mathbb{R}^3, x^2 \in \mathbb{R}^2$ , and

$$f(x) = \begin{bmatrix} x_2^1 + x_1^1 \\ x_3^1 + \cos(x_2^2) \\ x_1^1 + \sin(x_2^2) + \cos(x_3^1) \\ x_2^2 \\ 100 \arctan(x_2^1) - 100x_1^1 - 20x_2^2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

To apply Proposition 3.9, one computes  $\xi(x) = [x_1^2, \dots, L_f^4 x_1^2]^\top$  and the jacobian matrix  $O_z = \partial \xi(x) / \partial x$ . Since  $\det(O_z) = 1 / (1 + (x_2^1)^2)^3 > 0$ ,  $\xi(x)$  is a global diffeomorphism. The dimension of  $z$ -dynamics is five. Conditions (ii)-(iii) in Proposition 3.9 can be readily validated. The  $z$ -dynamics are equivalent to system (23), and the homogenous part is unstable. Due to the presence of terms  $\sin(x_2^2), \cos(x_2^1 x_3^1)$ , and  $\arctan(x_2^1)$ , the  $z$ -dynamics are not transformable to the observer form (19). Next, we try to apply Proposition 3.13. With

$$\begin{aligned} \xi^1 &= [x_1^2, L_f x_1^2]^\top, \\ \xi^2 &= [x_1^1, L_f x_1^1, L_f^2 x_1^1]^\top, \end{aligned}$$

the system representation in the new coordinates is

$$\dot{\xi}^1 = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} \xi^1 + \begin{bmatrix} 0 \\ 100 \arctan(\xi_2^2 - \xi_1^2) \end{bmatrix} + M^1 u,$$

$$\dot{\xi}^2 = \begin{bmatrix} x_2^2 \\ \xi_3^2 \\ \varphi(\xi) \end{bmatrix} + M^2 u,$$

$$y = \xi_1^2,$$

$$z = \xi_1^1,$$

where  $M^1$  and  $M^2$  are constant, and

$$\begin{aligned} \varphi &= 100 \sin(\xi_2^1) (\xi_1^1 - \arctan(\xi_2^2 - \xi_1^2)) + \xi_1^2 + \xi_3^2 \\ &\quad + 20 \sin(\xi_2^1) \xi_2^1 + \sin(\xi_2^1) + \cos(\xi_2^2 + \cos(\xi_2^1) - \xi_3^2). \end{aligned}$$

It is clear that both  $\xi^2$ -dynamics and  $\xi^1$ -dynamics are globally Lipschitz. A functional observer can be designed as given by (21), and the resultant error dynamics are GES, as shown in Proposition 3.14. Simulation is performed, in  $\xi$ -coordinates, to validate the design. Initial conditions of the original system and the observer are

$$\xi = [1, 1, 1, 1, 1]^\top, \quad \hat{\xi} = -\xi.$$

The observer gain is  $SC_y^{-1} = [3\theta, 3\theta^2, \theta^3]^\top$ . Results are shown in Figs. 5-8, where Figs. 5-6 correspond to the case  $\theta = 4$ , and Figs. 7-8 for  $\theta = 20$ . The observer results in convergent estimation error dynamics for both cases, albeit convergent rates are different. The homogenous part of  $\tilde{\xi}^1$ -dynamics has poles at  $-10$ . With  $\theta = 4$ ,  $\tilde{\xi}^2$  converges slowly, and dominates the  $\tilde{\xi}^1$ -dynamics. On the contrary, with  $\theta = 20$ ,  $\tilde{\xi}^2$  converges faster than  $\tilde{\xi}^1$ , which is dominated by its homogenous part.

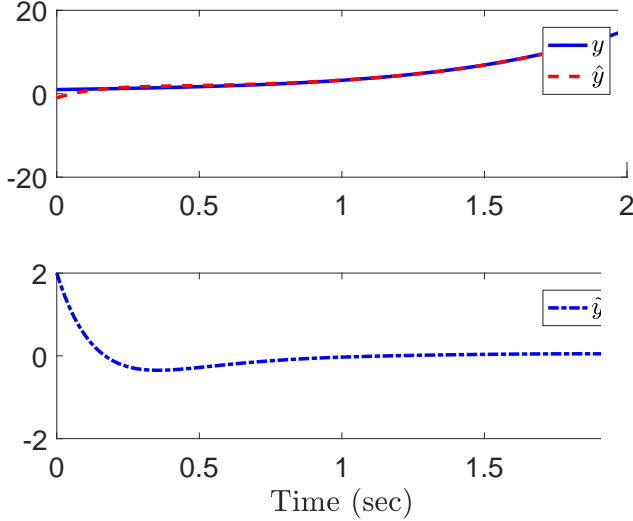


Fig. 5:  $y(t), \hat{y}(t), \tilde{y}(t)$  for  $\theta = 4$

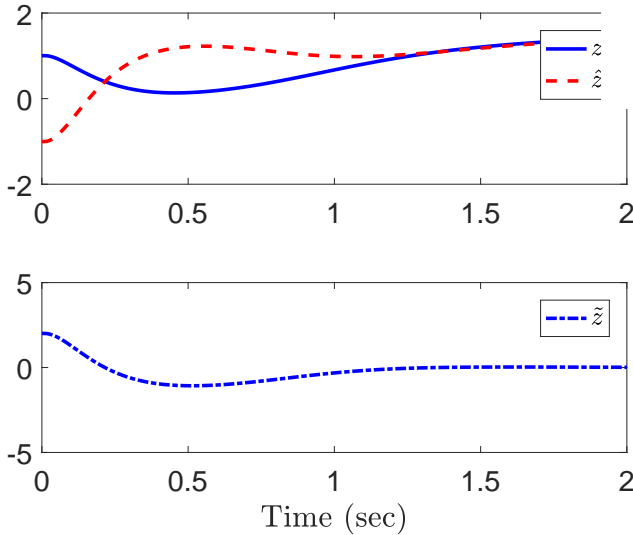


Fig. 6:  $z(t), \hat{z}(t), \tilde{z}(t)$  for  $\theta = 4$

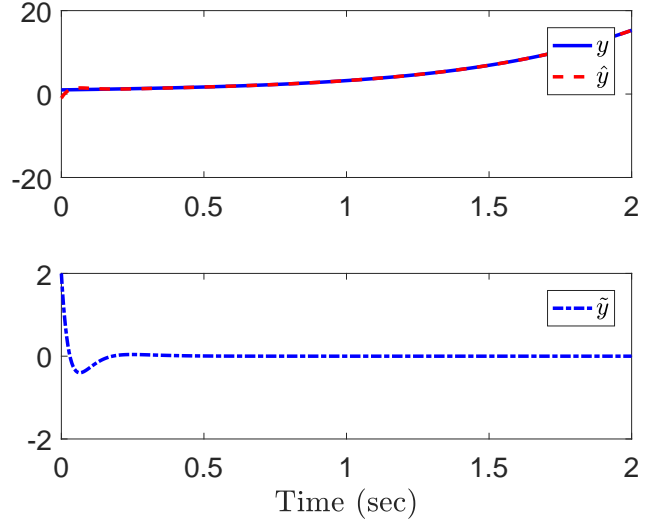


Fig. 7:  $y(t), \hat{y}(t), \tilde{y}(t)$  for  $\theta = 20$

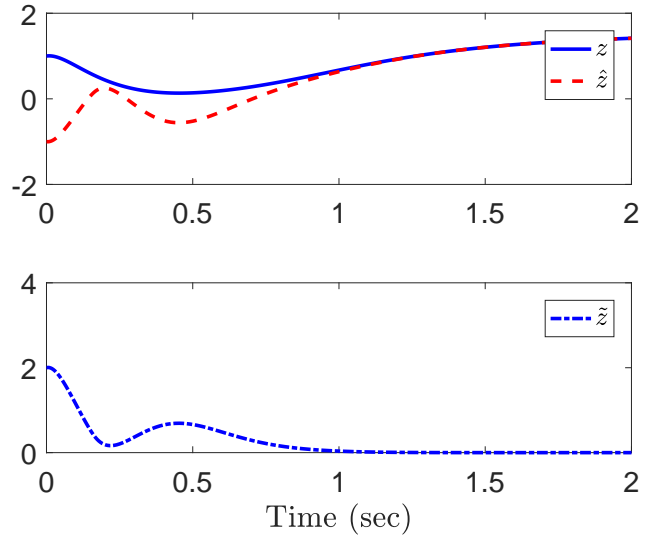


Fig. 8:  $z(t), \hat{z}(t), \tilde{z}(t)$  for  $\theta = 20$

## 5 Conclusion

This work began with the acceleration-based control of a semi-active vibration reduction system, where the reconstruction of the relative velocity is tackled by function observer design. The idea was further generalized to investigate function estimation problem for a class of single input single output linear and nonlinear systems. Specifically, necessary and sufficient existence conditions of linear functional observer for LTI systems were established. For nonlinear system, existence conditions of nonlinear functional observers were established. The proposed designs were validated by simulation.

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