MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

Quasi-Newton Jacobian and Hessian Updates for Pseudospectral based NMPC

Quirynen, R.; Hespanhol, P.

TR2018-119 August 25, 2018

Abstract

Pseudospectral and collocation methods form a popular direct approach to solving continuous-time optimal control problems. Lifted Newton-type algorithms have been proposed as a computationally efficient way to implement online pseudospectral methods for nonlinear model predictive control (NMPC). The present paper extends this work based on a rank-one Jacobian update formula for the nonlinear system dynamics. In addition, we describe an algorithm implementation where this rank-one Jacobian update can be used directly to compute a low-rank update to the condensed Hessian, resulting in an overall quadratic computational complexity for each iteration. A preliminary C code implementation is shown to allow considerable numerical speedups for the optimal control case study of the nonlinear chain of masses.

IFAC Conference on Nonlinear Model Predictive Control

Quasi-Newton Jacobian and Hessian Updates for Pseudospectral based NMPC

Pedro Hespanhol [*,**] **Rien Quirynen** [*]

[*] *Mechatronics, Mitsubishi Electric Research Laboratories, Cambridge, MA, 02139, USA. (e-mail: quirynen@merl.com).*
[**] *Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, USA.*

Abstract: Pseudospectral and collocation methods form a popular direct approach to solving continuous-time optimal control problems. Lifted Newton-type algorithms have been proposed as a computationally efficient way to implement online pseudospectral methods for nonlinear model predictive control (NMPC). The present paper extends this work based on a rank-one Jacobian update formula for the nonlinear system dynamics. In addition, we describe an algorithm implementation where this rank-one Jacobian update can be used directly to compute a low-rank update to the condensed Hessian, resulting in an overall quadratic computational complexity for each iteration. A preliminary C code implementation is shown to allow considerable numerical speedups for the optimal control case study of the nonlinear chain of masses.

Keywords: Nonlinear predictive control, quasi-Newton methods, Optimal control

1. INTRODUCTION

There has been an increasing interest in using dynamic optimization for real-time applications, i.e., in the context of model predictive control (MPC) and moving horizon estimation (MHE) (Rawlings et al., 2017). For this purpose, an optimal control problem (OCP) needs to be solved at each time instant, under strict timing constraints. Tailored continuation based online optimization algorithms have been developed for real-time optimal control as discussed in (Diehl et al., 2009). A popular example is the real-time iteration (RTI) algorithm (Diehl et al., 2005), an online variant of sequential quadratic programming (SQP) for nonlinear MPC (NMPC) applications.

This article aims at solving the following OCP formulation in continuous time

$$\begin{aligned}
\min_{x(\cdot),\,u(\cdot)} \quad & \int_0^T \|F(x(t),u(t))\|_2^2 \,\mathrm{d}t && && \text{(1a)}\\
\text{s.t.} \quad & 0 = x(0) - \hat{x}_0, && && \text{(1b)}\\
& 0 = f(\dot{x}(t), x(t), u(t)), && \forall t \in [0,T], && \text{(1c)}\\
& 0 \geq h(x(t), u(t)), && \forall t \in [0,T], && \text{(1d)}\\
& 0 \geq r(x(T)), && && \text{(1e)}
\end{aligned}$$

where $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ denote the differential states and $u(t) \in \mathbb{R}^{n_\mathrm{u}}$ are the control inputs at time t. The objective in Eq. (1a) consists of a nonlinear least squares type Lagrange term. The problem depends on the parameter value $\hat{x}_0$ through the initial condition of Eq. (1b). The nonlinear dynamics in Eq. (1c) are described by an implicit system of ordinary differential equations (ODE), even though this can generally be extended to differential-algebraic equations (DAE) of index 1. Respectively, Eqs. (1d) and (1e) denote the path and terminal inequality constraints.

A popular direct technique for solving the optimal control problem in (1) is based on orthogonal collocation, where a distinction is made between the use of *local* and *global* collocation polynomials (Rao, 2010). In local collocation, also referred to as direct collocation (Betts, 2010; Biegler, 1984), one uses piecewise polynomials which are typically of a fixed degree. Pseudospectral methods form an extreme case of such an approach, by mainly increasing or decreasing the degree of a global collocation polynomial. Given a smooth and well-behaved optimal control solution, this approximation is known to converge at an exponential rate (Rao, 2010). Another reason for their popularity is that any collocation method can readily be applied to problems involving stiff or implicit systems of differential equations. Orthogonal collocation methods are typically used, based on the roots of Chebyshev or Legendre polynomials. We focus on Legendre collocation methods, which employ a quadrature rule based on either Gauss, Radau or Lobatto points (Hairer and Wanner, 1991).

It has been shown how collocation schemes can be used within a lifted Newton-type implementation, which bridges the gap between direct collocation and direct multiple shooting (Bock and Plitt, 1984) as discussed in (Quirynen et al., 2017). Recently, a lifted Newton-type optimization algorithm for pseudospectral based NMPC has been proposed in (Quirynen and Diehl, 2018), based on a tailored Jacobian approximation technique and the inexact Newton method with iterated sensitivities (INIS) from (Quirynen et al., 2018a). In addition, adjoint based quasi-Newton Jacobian update schemes for constrained optimization (Diehl et al., 2010; Griewank and Walther, 2002) have effectively been applied to the lifted collocation algorithm in (Hespanhol and Quirynen, 2018).

The present paper extends both the work in (Quirynen and Diehl, 2018) and in (Hespanhol and Quirynen, 2018) by proposing a tailored quasi-Newton type Jacobian and Hessian update scheme with numerical condensing

and expansion of the collocation variables, resulting in a pseudospectral based NMPC algorithm with an overall quadratic computational complexity.

The paper is organized as follows. Section 2 summarizes collocation schemes and their use in direct optimal control methods. Section 3 introduces the lifted Newton implementation of a pseudospectral method with quasi-Newton Jacobian updates. The corresponding low-rank Hessian update schemes are presented in Section 4. The proposed algorithms are illustrated based on numerical results of NMPC for the chain of masses in Section 5.

2. DIRECT OPTIMAL CONTROL METHODS

Direct optimal control (Bock and Plitt, 1984) tackles the continuous time OCP (1) by forming a discrete time approximation and solving the resulting NLP.

2.1 Collocation based Numerical Simulation

In order to arrive at a compact notation, we consider a collocation polynomial of degree N for the parametrization of both the state and control profile. Let us define the time transformation $\tau := \frac{t}{T}$, such that $\tau \in [0,1]$ for $t \in [0,T]$. The polynomial approximation for the differential state can then be obtained as follows

$$p_x(c) = x_0 + T\sum_{i=1}^{N} \mathrm{k}_i \int_0^c \ell_i(\tau)\,\mathrm{d}\tau, \tag{2}$$

where $\ell_i(\tau)$ denote the Lagrange interpolating polynomials, given a set of collocation nodes $0 \leq c_i \leq 1$ for $i = 1,\dots,N$ and the corresponding stage values k_i and u_i, respectively, for the state derivatives and the control inputs. Note that the parametrized control profile reads as $p_u(c) = \sum_{i=1}^{N} \ell_i(c) u_i$ such that $p_u(c_i) = u_i$, for $i = 1,\dots,N$. The collocation variables k_i are defined by imposing the system dynamics in Eq. (1c):

$$G(x_0, U, K) = \begin{bmatrix} f(\mathrm{k}_1, x_0 + T\sum_{j=1}^{N} a_{1j}\mathrm{k}_j, u_1) \\ \vdots \\ f(\mathrm{k}_N, x_0 + T\sum_{j=1}^{N} a_{Nj}\mathrm{k}_j, u_N) \end{bmatrix} = 0, \tag{3}$$

which denotes the nonlinear system of collocation equations and where $a_{ij} = \int_0^{c_i} \ell_j(\tau)\,\mathrm{d}\tau$ is defined. The numerical simulation result at the end of the interval reads as

$$x(T) \approx x_T(K) = x_0 + T\sum_{i=1}^{N} b_i \mathrm{k}_i = p_x(1), \tag{4}$$

where $b_i = \int_0^1 \ell_i(\tau)\,\mathrm{d}\tau$. All collocation schemes belong to the family of implicit Runge-Kutta (IRK) methods, which are often defined based on their Butcher tableau.

2.2 Pseudospectral Optimal Control

Based on the same Gaussian quadrature rule as used for the collocation scheme in (4), let us define a discretization for the least squares type objective in (1a):

$$\int_0^T \|F(x(t),u(t))\|_2^2 \approx T\sum_{i=1}^{N} b_i \|F(x_i,u_i)\|_2^2, \tag{5}$$

where $x_i(K) = x_0 + T\sum_{j=1}^{N} a_{ij}\mathrm{k}_j$. Direct transcription, of which pseudospectral methods form a special subclass, is then based on including the additional variables and equations (3) directly into the discrete time OCP formulation. Based on the discretized cost and by imposing the path constraints in (1d) at the collocation nodes, the resulting dense nonlinear program (NLP) reads as

$$\begin{aligned}
\min_{x_0,U,K}\ & T\sum_{i=1}^{N} b_i \|F(x_i(K),u_i)\|_2^2 && \text{(6a)}\\
\text{s.t.}\ & 0 = x_0 - \hat{x}_0, && \text{(6b)}\\
& 0 = f(\mathrm{k}_i, x_i(K), u_i), \quad i = 1,\dots,N, && \text{(6c)}\\
& 0 \geq h(x_i(K), u_i), \quad i = 1,\dots,N, && \text{(6d)}\\
& 0 \geq r(x_T(K)), && \text{(6e)}
\end{aligned}$$

where the stage values $U = [u_1^\top,\dots,u_N^\top]^\top \in \mathbb{R}^{Nn_\mathrm{u}}$ and $K = [\mathrm{k}_1^\top,\dots,\mathrm{k}_N^\top]^\top \in \mathbb{R}^{Nn_\mathrm{x}}$ are defined.

2.3 Gauss-Newton based SQP Method

An adjoint based Gauss-Newton SQP algorithm for the NLP in (6) relies on the solution of a quadratic program (QP) approximation in each iteration:

$$\begin{aligned}
\min_{\Delta U,\Delta K}\ & T\sum_{i=1}^{N} b_i \|F_i + J_i^x \Delta x_i + J_i^u \Delta u_i\|_2^2 && \text{(7a)}\\
& + \omega^{\mathrm{o}\top}\left[\left(\frac{\partial G}{\partial U} - D\right)\left(\frac{\partial G}{\partial K} - C\right)\right]\begin{bmatrix}\Delta U\\ \Delta K\end{bmatrix} && \text{(7b)}\\
\text{s.t.}\ & 0 = g + D\Delta U + C\Delta K, && \text{(7c)}\\
& 0 \geq a + A_u\Delta U + A_\mathrm{k}\Delta K, && \text{(7d)}
\end{aligned}$$

where $F_i := F(\hat{x}_0 + T\sum_{j=1}^{N} a_{ij}\mathrm{k}_j^\mathrm{o}, u_i^\mathrm{o})$, $\Delta x_i = T\sum_{j=1}^{N} a_{ij}\Delta\mathrm{k}_j$ and $g := G(\hat{x}_0, U^\mathrm{o}, K^\mathrm{o})$ are defined. Note that the initial state variable $x_0 = \hat{x}_0$ has been eliminated to arrive at a more compact notation. The values $U^\mathrm{o}, K^\mathrm{o}$ denote the linearization point and ω^o denotes the values for the Lagrange multipliers $\omega \in \mathbb{R}^{Nn_\mathrm{x}}$ corresponding to the collocation equations in (6c), based on the previous SQP iteration. The Jacobian matrices read $J_i^x = \frac{\partial F_i}{\partial x_i}$ and $J_i^u = \frac{\partial F_i}{\partial u_i}$ for the objective. The constraint Jacobian approximations $D \approx \frac{\partial G}{\partial U}(\cdot)$ and $C \approx \frac{\partial G}{\partial K}(\cdot)$ will be discussed further in Section 3. Given these constraint Jacobian approximations, the adjoint gradient correction for the inexact SQP method (Wirsching et al., 2006) is defined in Eq. (7b). The inequality constraints in (7d) denote an exact linearization of the path and terminal constraints in (6d) and (6e), which is typically relatively cheap to evaluate.

In embedded NMPC applications, one needs to solve the nonlinear OCP of Eq. (6) at each sampling instant under strict timing constraints. For this purpose, we instead use the real-time iteration (RTI) scheme (Diehl et al., 2009, 2005) for nonlinear MPC, which is a continuation based variant of a fixed-step SQP method. More specifically, by warm-starting the algorithm based on the (approximate) solution to the OCP at a previous time instant, only one QP subproblem of the form in (7) needs to be solved at each time step. The general idea is that one prefers to obtain new measurement information from the system, rather than iterating until convergence for an optimization problem that is becoming outdated.

3. LIFTED NEWTON-TYPE OPTIMIZATION WITH RANK-ONE JACOBIAN UPDATES

Let us describe the proposed lifted Newton-type optimization algorithm for pseudospectral based NMPC, using a quasi-Newton type rank-one Jacobian update formula.

3.1 Lifted Newton-Type Optimization

An efficient way to solve the QP subproblem in (7) is based on the combination of condensing and expansion. This corresponds to a numerical elimination of the collocation variables, by defining the following quantities

$$\Delta\tilde{K} = -C^{-1}g \quad \text{and} \quad E = -C^{-1}D, \tag{8}$$

such that $\Delta K = \Delta\tilde{K} + E\Delta U$. Based on the inexact Newton step in (8), the subproblem can be reformulated as the following dense QP

$$\min_{\Delta U} \quad \frac{1}{2}\Delta U^\top H_\mathrm{c}\,\Delta U + h_\mathrm{c}^\top \Delta U \tag{9a}$$
$$\text{s.t.} \quad 0 \geq a_\mathrm{c} + A_\mathrm{c}\Delta U, \tag{9b}$$

where the vectors $a_\mathrm{c} = a + A_\mathrm{k}\Delta\tilde{K}$ and $h_\mathrm{c} = \begin{bmatrix}\mathbb{1} & E^\top\end{bmatrix} h$ are defined and the condensed matrices read as $A_\mathrm{c} = A_u + A_\mathrm{k}E$ and $H_\mathrm{c} = \begin{bmatrix}\mathbb{1} & E^\top\end{bmatrix} H \begin{bmatrix}\mathbb{1} \\ E\end{bmatrix}$. The condensed Gauss-Newton based objective is defined, using the Hessian matrix H and gradient vector h for the objective function in Eq. (7a) including the gradient correction in Eq. (7b).

Based on the solution of the condensed QP subproblem in (9), the inexact Newton (IN) method requires the additional computation of the Lagrange multipliers ω corresponding to the collocation equations in (6c). We use λ to denote the Lagrange multipliers for the inequality constraints in (9b) or (7d) and λ^o, λ^+ denote, respectively, the values in the previous and current SQP iteration. Based on the optimality conditions for the QP in Eq. (7), using the Jacobian approximation $C \approx \frac{\partial G}{\partial K}(\cdot)$, this results in the Newton-type update for the ω multipliers:

$$\Delta\omega = -C^{-\top}\left(h_\mathrm{k} + \frac{\partial G}{\partial K}^\top \omega^\mathrm{o} + A_\mathrm{k}^\top \lambda^+\right), \tag{10}$$

where $h_\mathrm{k} \in \mathbb{R}^{N n_\mathrm{x}}$ denotes the gradient of the QP objective term in (7a) with respect to the collocation variables. The updated multiplier values read as $\omega^+ = \omega^\mathrm{o} + \Delta\omega$ and the collocation variables are updated as follows $K^+ = K^\mathrm{o} + \Delta\tilde{K} + E\Delta U$, given the multiplier values λ^+ and solution vector $\Delta U^\star$ from solving the dense QP (9).

3.2 Quasi-Newton Jacobian Update Formula

Unlike standard Broyden type methods (Broyden, 1967), a two-sided rank-one (TR1) update formula has been proposed in (Diehl et al., 2010; Griewank and Walther, 2002) as a generalization of the symmetric rank-one (SR1) update scheme in (Conn et al., 1991) for constrained optimization. The TR1 formula enjoys several benefits over classical methods, such as heredity and linear transformation invariance (Griewank and Walther, 2002).

Let us apply the TR1 update formula to the Jacobian approximation $[D\ C] \approx \frac{\partial G(\cdot)}{\partial (U,K)}$. The key ingredient of the TR1 method is that it aims to simultaneously satisfy the *direct* secant condition $\begin{bmatrix}D^+ & C^+\end{bmatrix} s = y$ and the *adjoint* or *transposed* secant condition $\sigma^\top \begin{bmatrix}D^+ & C^+\end{bmatrix} = \mu^\top$, where we define the adjoint $\mu^\top = \sigma^\top \frac{\partial G}{\partial (U,K)}(\hat{x}_0, U^+, K^+)$, given $\sigma = \omega^+ - \omega^\mathrm{o}$, the difference in function evaluations $y = G(\hat{x}_0, U^+, K^+) - G(\hat{x}_0, U^\mathrm{o}, K^\mathrm{o})$ and $s := \begin{bmatrix}U^+ - U^\mathrm{o} \\ K^+ - K^\mathrm{o}\end{bmatrix}$. Note that the gradient $\sigma^\top \frac{\partial G}{\partial (U,K)}(\cdot)$ can be computed efficiently using the backward mode of algorithmic differentiation (AD) (Griewank, 2000). The TR1 based Jacobian update formula then reads as

$$[D^+\ C^+] = [D^\mathrm{o}\ C^\mathrm{o}] + \alpha\,(y - [D^\mathrm{o}\ C^\mathrm{o}]s)\left(\mu^\top - \sigma^\top [D^\mathrm{o}\ C^\mathrm{o}]\right), \tag{11}$$

where the scalar α can be defined for different variants of the update scheme. Aside from the case where the function $G(\cdot)$ is affine, the two secant conditions are not consistent with each other and they can therefore not both be satisfied by the updated matrix $\begin{bmatrix}D^+ & C^+\end{bmatrix}$. In the adjoint variant of the TR1 update, the value $\alpha_\mathrm{A} = 1/(\sigma^\top y - \sigma^\top [D^\mathrm{o}\ C^\mathrm{o}]s)$ is defined such that the adjoint secant condition is satisfied exactly and the forward condition holds up to some accuracy. Similarly, the forward variant is based on $\alpha_\mathrm{F} = 1/(\mu^\top s - \sigma^\top [D^\mathrm{o}\ C^\mathrm{o}]s)$ and instead satisfies the direct secant condition exactly.

3.3 Real-Time Iteration Scheme for NMPC

In order to use the TR1 Jacobian update formula (11) within a lifted Newton-type optimization algorithm, one needs to be able to efficiently form the condensed QP in (9). For this purpose, the work in (Hespanhol and Quirynen, 2018) described how to directly update the condensed matrix $E = -C^{-1}D$ in Eq. (8) from one SQP iteration to the next. Let us write the rank-one update formula from Eq. (11) as follows

$$D^+ = D^\mathrm{o} + \alpha\, u v_\mathrm{D}^\top \quad \text{and} \quad C^+ = C^\mathrm{o} + \alpha\, u v_\mathrm{C}^\top, \tag{12}$$

where $u = y - [D^\mathrm{o}\ C^\mathrm{o}]s$ and $[v_\mathrm{D}^\top\ v_\mathrm{C}^\top] = \mu^\top - \sigma^\top [D^\mathrm{o}\ C^\mathrm{o}]$. The Sherman-Morrison formula can be used to update the matrix inverse approximation $C^{\mathrm{o}^{-1}} \approx \frac{\partial G}{\partial K}^{-1}$ as

$$C^{+^{-1}} = C^{\mathrm{o}^{-1}} - \alpha\beta\, u_1 v_\mathrm{C}^\top C^{\mathrm{o}^{-1}}, \tag{13}$$

where $u_1 = C^{\mathrm{o}^{-1}} u$ and $\beta = \frac{1}{1+\alpha v_\mathrm{C}^\top u_1}$. It can be shown that the rank-one update formula then reads as

$$E^+ = -C^{+^{-1}} D^+ = E^\mathrm{o} + u_1 v_1^\top, \tag{14}$$

where $v_1^\top = \alpha\beta v_\mathrm{C}^\top (E^\mathrm{o} + \alpha u_1 v_\mathrm{D}^\top) - \alpha v_\mathrm{D}^\top$.

This rank-one update for the matrix $E^+ = -C^{+^{-1}} D^+$ in Eq. (14) provides an efficient manner to directly compute the matrices in the condensed QP (9), without the need for a matrix factorization, matrix inversion and without any matrix-matrix multiplications. Instead, the proposed algorithm merely requires matrix-vector multiplications and outer products, resulting in an overall quadratic $\mathcal{O}\left(N^2 m^2\right)$ instead of cubic $\mathcal{O}\left(N^3 m^3\right)$ computational complexity, where $m = (n_\mathrm{x} + n_\mathrm{u})$ denotes the number of state and control variables. The resulting implementation of the RTI scheme with TR1 based Jacobian updates for pseudospectral based NMPC is presented in Algorithm 1.

Remark 1. The new input value can be applied to the controlled system in step 4 of Alg. 1. A pseudospectral

Algorithm 1 Pseudospectral Method with TR1 Jacobian Updates within a Real-Time Iteration Scheme for NMPC.

Input: U^{o}, K^{o}, λ^{o}, ω^{o}, C^{o}, D^{o}, $C^{\mathrm{o}^{-1}}$ and E^{o}.

`Problem linearization`

Formulate the dense QP in (9) with $A_{\mathrm{c}} = A_u + A_{\mathrm{k}} E^{\mathrm{o}}$ and condensed Hessian $H_{\mathrm{c}} = E^{\mathrm{o}^\top} H E^{\mathrm{o}}$.

`Computation of step direction`

Obtain current state estimate $\hat{x}_0$. ▷ from system

Evaluate the vectors $a_{\mathrm{c}}, h_{\mathrm{c}}$ and solve the QP (9): $U^+ \leftarrow U^{\mathrm{o}} + \Delta U^\star$ and $\lambda^+ \leftarrow \lambda^\star$.

Apply new control input value. ▷ to system

`TR1 Jacobian update`

$K^+ \leftarrow K^{\mathrm{o}} + \Delta\tilde{K} + E^{\mathrm{o}} \Delta U^\star$,

$\omega^+ \leftarrow \omega^{\mathrm{o}} - C^{-\top} \left(h_{\mathrm{k}} + \frac{\partial G}{\partial K}^\top \omega^{\mathrm{o}} + A_{\mathrm{k}}^\top \lambda^+ \right)$,

$D^+ \leftarrow D^{\mathrm{o}} + \alpha u v_{\mathrm{D}}^\top$ and $C^+ \leftarrow C^{\mathrm{o}} + \alpha u v_{\mathrm{C}}^\top$,

$C^{+^{-1}} \leftarrow C^{\mathrm{o}^{-1}} - \alpha\beta\, u_1 v_{\mathrm{C}}^\top C^{\mathrm{o}^{-1}}$ and $E^+ \leftarrow E^{\mathrm{o}} + u_1 v_1^\top$.

Output: U^+, K^+, λ^+, ω^+, C^+, D^+, $C^{+^{-1}}$ and E^+.

method provides a continuous time control profile that is represented by the polynomial $p_u(c) = \sum_{i=1}^{N} \ell_i(c) u_i$. This continuous time trajectory can more or less accurately be applied to the system, depending on the particular actuation in the control application and its sampling frequency. For simplicity, let us further assume a piecewise constant actuation, where we use the value $p_u(\frac{T_{\mathrm{s}}}{T})$ of the collocation polynomial in which T_{s} denotes the MPC sampling time and T the control horizon length.

4. QUASI-NEWTON TYPE UPDATE SCHEME FOR THE CONDENSED HESSIAN

The TR1 Jacobian update scheme has a quadratic computational complexity of $\mathcal{O}\left(N^2 m^2\right)$. Constructing the condensed Hessian contribution $H_{\mathrm{c}} = E^\top H E$ and computing a matrix factorization or inverse however requires a cubic computational complexity of $\mathcal{O}\left(N^3 m^3\right)$ in general. Given the condensed Hessian and its inverse or matrix decomposition, the runtime computational cost for solving the dense QP (9) can be made of quadratic complexity $\mathcal{O}\left(N^2 m^2\right)$ instead, e.g., using a dense variant of the active-set method from (Quirynen et al., 2018b). Let us focus on how to avoid the operations with cubic complexity in case of a constant Hessian approximation or when using a quasi-Newton type update scheme.

4.1 Constant Hessian Approximation: Gauss-Newton

Note that the Gauss-Newton type Hessian approximation in Eq. (7a) corresponds to a constant matrix H, in case of a quadratic objective (6a) in the original NLP formulation. This is rather common in practical applications of MPC when tracking a reference for a linear output function of the state and control variables. Let us look at the condensed Hessian H_{c}, given the constant matrix H and a rank-one Jacobian update as in (14).

Lemma 2. (SR2). Given a rank-one update to the condensed Jacobian $E^+ = E^{\mathrm{o}} + u_1 v_1^\top$, the condensed Hessian matrix $H_{\mathrm{c}} = E^\top H E$ can be computed using the following symmetric rank-two update:

$$H_{\mathrm{c}}^+ = H_{\mathrm{c}}^{\mathrm{o}} + \tilde{u}_1 v_1^\top + v_1 \left(\tilde{u}_1^\top + \beta_1\, v_1^\top \right). \tag{15}$$

Proof. This follows directly from the expression for the updated condensed Hessian matrix

$$\begin{aligned} H_{\mathrm{c}}^+ &= E^{+^\top} H E^+ = \left(E^{\mathrm{o}} + u_1 v_1^\top\right)^\top H \left(E^{\mathrm{o}} + u_1 v_1^\top\right) \\ &= H_{\mathrm{c}}^{\mathrm{o}} + E^{\mathrm{o}^\top} H u_1 v_1^\top + v_1 u_1^\top H E^{\mathrm{o}} + v_1 u_1^\top H u_1 v_1^\top \\ &= H_{\mathrm{c}}^{\mathrm{o}} + \tilde{u}_1 v_1^\top + v_1 \left(\tilde{u}_1^\top + \beta_1\, v_1^\top\right), \end{aligned} \tag{16}$$

where $\beta_1 := u_1^\top H u_1$ and $\tilde{u}_1 := E^{\mathrm{o}^\top} H u_1$, such that the symmetric update is readily identified to be of rank 2. □

Note that the symmetric rank-two (SR2) update (15) can alternatively be represented as follows:

$$H_{\mathrm{c}}^+ = H_{\mathrm{c}}^{\mathrm{o}} + \left(\frac{1}{\tilde{\beta}_1}\tilde{u}_1 + \tilde{\beta}_1 v_1\right)\left(\frac{1}{\tilde{\beta}_1}\tilde{u}_1 + \tilde{\beta}_1 v_1\right)^\top - \frac{1}{\beta_1}\tilde{u}_1\tilde{u}_1^\top, \tag{17}$$

where $\tilde{\beta}_1 := \sqrt{\beta_1}$ given that $\beta_1 = u_1^\top H u_1 > 0$. This means that the condensed Hessian matrix can be updated, from one SQP iteration to the next, using the SR2 update or using two consecutive symmetric rank-one updates as in Eq. (17). Similarly, the Cholesky factorization, or the matrix inverse using the Sherman-Morrison-Woodbury formula, can be updated directly for the condensed Hessian. The resulting algorithm implementation, with overall quadratic computational complexity based on the TR1 and SR2 update formulas, for pseudospectral based nonlinear MPC is presented in Algorithm 2.

4.2 Quasi-Newton Type Hessian Approximation

We can construct a similar update formula for the condensed Hessian in case that a quasi-Newton type method is used instead of a constant Hessian approximation. For simplicity, let us consider the symmetric rank-one (SR1) update formula (Conn et al., 1991) to approximate the Hessian of the Lagrangian. This results in the STR1 update procedure as described in (Diehl et al., 2010).

Lemma 3. (SR3). Given a rank-one update to the condensed Jacobian $E^+ = E^{\mathrm{o}} + u_1 v_1^\top$ and a symmetric rank-one Hessian update $H^+ = H^{\mathrm{o}} + \alpha_2\, u_2 u_2^\top$, the condensed Hessian matrix $H_{\mathrm{c}} = E^\top H E$ can be computed using the symmetric rank-three update:

$$\begin{aligned} H_{\mathrm{c}}^+ = H_{\mathrm{c}}^{\mathrm{o}} &+ \alpha_2\, \tilde{u}_2 \tilde{u}_2^\top + v_1 \left(\tilde{u}_1^\top + \beta_3\, \tilde{u}_2^\top + \beta_4\, v_1^\top\right) \\ &+ \left(\tilde{u}_1 + \beta_3\, \tilde{u}_2 + \beta_4\, v_1\right) v_1^\top. \end{aligned} \tag{18}$$

Proof. It follows from the expression for the updated condensed Hessian matrix

$$\begin{aligned} H_{\mathrm{c}}^+ &= E^{+^\top} H^+ E^+ \\ &= \left(E^{\mathrm{o}} + u_1 v_1^\top\right)^\top \left(H^{\mathrm{o}} + \alpha_2\, u_2 u_2^\top\right) \left(E^{\mathrm{o}} + u_1 v_1^\top\right) \\ &= H_{\mathrm{c}}^{\mathrm{o}} + \tilde{u}_1 v_1^\top + v_1 \tilde{u}_1^\top + \beta_1\, v_1 v_1^\top + \alpha_2\, \tilde{u}_2 \tilde{u}_2^\top \\ &\quad + \alpha_2\, \beta_2\, \tilde{u}_2 v_1^\top + \alpha_2\, \beta_2\, v_1 \tilde{u}_2^\top + \alpha_2\, \beta_2^2\, v_1 v_1^\top, \end{aligned} \tag{19}$$

where $\beta_1 := u_1^\top H u_1$, $\beta_2 := u_2^\top u_1$, $\tilde{u}_1 := E^{\mathrm{o}^\top} H u_1$ and $\tilde{u}_2 := E^{\mathrm{o}^\top} u_2$. This can be further simplified to

$$\begin{aligned} H_{\mathrm{c}}^+ = H_{\mathrm{c}}^{\mathrm{o}} &+ \alpha_2\, \tilde{u}_2 \tilde{u}_2^\top + v_1 \left(\tilde{u}_1^\top + \beta_3\, \tilde{u}_2^\top + \beta_4\, v_1^\top\right) \\ &+ \left(\tilde{u}_1 + \beta_3\, \tilde{u}_2 + \beta_4\, v_1\right) v_1^\top, \end{aligned} \tag{20}$$

where $\beta_3 := \alpha_2\, \beta_2$ and $\beta_4 := \frac{\beta_1 + \alpha_2\, \beta_2^2}{2}$, such that the symmetric update is readily identified to be of rank 3. □

Algorithm 2 Pseudospectral Method with TR1 Jacobian and SR2/SR3 condensed Hessian Updates for NMPC.

Input: U^{o}, K^{o}, λ^{o}, ω^{o}, C^{o}, D^{o}, $C^{\mathrm{o}^{-1}}$, E^{o}, H^{o} and $H_{\mathrm{c}}^{\mathrm{o}}$.

`Problem linearization`

1: Formulate the dense QP in (9) with A_{c} and $H_{\mathrm{c}}^{\mathrm{o}}$.

2: `Computation of step direction`: line 2-4 in Alg. 1

3: `TR1 Jacobian update`: line 5-8 in Alg. 1

Option 1: `SR2 condensed Hessian update`

4: $H^{+} \leftarrow H^{\mathrm{o}}$.

5: $H_{\mathrm{c}}^{+} \leftarrow H_{\mathrm{c}}^{\mathrm{o}} + \tilde{u}_1 v_1^{\top} + v_1 \left(\tilde{u}_1^{\top} + \beta_1 \, v_1^{\top}\right)$.

Option 2: `SR3 condensed Hessian update`

6: $H^{+} \leftarrow H^{\mathrm{o}} + \alpha_2 \, u_2 u_2^{\top}$

7: $H_{\mathrm{c}}^{+} \leftarrow H_{\mathrm{c}}^{\mathrm{o}} + \alpha_2 \, \tilde{u}_2 \tilde{u}_2^{\top} + v_1 \left(\tilde{u}_1^{\top} + \beta_3 \, \tilde{u}_2^{\top} + \beta_4 \, v_1^{\top}\right) + \left(\tilde{u}_1 + \beta_3 \, \tilde{u}_2 + \beta_4 \, v_1\right) v_1^{\top}$.

Output: U^{+}, K^{+}, λ^{+}, ω^{+}, C^{+}, D^{+}, $C^{+^{-1}}$, E^{+}, H^{+}, H_{c}^{+}.

In a similar manner as for the symmetric rank-two update from Lemma 2, an alternative representation of the symmetric rank-three (SR3) formula (18) can be constructed as a sequence of three consecutive symmetric rank-one updates. In addition, the Cholesky factorization, or the matrix inverse using the Sherman-Morrison-Woodbury formula, can be computed directly for the condensed Hessian based on this update scheme. Algorithm 2 describes an implementation of pseudospectral based NMPC, using the TR1 and SR3 update formulas, respectively, for the condensed Jacobian and Hessian approximations.

5. NMPC CASE STUDY: CHAIN OF MASSES

We consider the chain mass problem as a benchmark example for nonlinear MPC, which allows one to intuitively change the number of masses and therefore the state dimension in the problem. The control task is to return a chain of n_{m} masses connected with springs to its steady state, starting from a perturbed initial configuration. The mass at one end is fixed, while the control input $u \in \mathbb{R}^3$ to the system is the direct force applied to the mass at the other end of the chain. This dynamic system can be described by a state vector $x \in \mathbb{R}^{6(n_{\mathrm{m}}-1)}$, which is governed by the set of nonlinear differential equations in (Quirynen et al., 2017; Wirsching et al., 2006).

Our aim is to validate the computational performance for Algorithm 1 and 2, using a lifted Newton-type optimization method with TR1 based Jacobian and corresponding condensed Hessian updates, in comparison with the standard RTI scheme based on exact Jacobian evaluations. The preliminary software implementation of the presented algorithms consists of C code for the TR1 and SR2 update formulas, in combination with a dense variant of the `PRESAS` active-set QP solver (Quirynen et al., 2018b) and code generated evaluations of the system dynamics and the adjoint derivatives using `CasADi` (Andersson, 2013). In addition, we include a comparison with the direct collocation based RTI scheme with block-TR1 Jacobian updates as presented in (Hespanhol and Quirynen, 2018).

5.1 Pseudospectral versus Direct Collocation Methods

The condensing procedure in a classical lifted Newton optimization algorithm for pseudospectral based optimal control requires a factorization of the exact Jacobian matrix at each iteration, resulting in a computational complexity of $\mathcal{O}\left(N^3 m^3\right)$. The proposed TR1 Jacobian update scheme with numerical condensing in Algorithm 1 avoids all cubic operations for the Jacobian approximation and Algorithm 2 additionally avoids such costly operations for the condensed Hessian, resulting in an overall computational complexity of $\mathcal{O}\left(N^2 m^2\right)$. The block-TR1 Jacobian update formula itself has a computational complexity of $\mathcal{O}\left(N_{\mathrm{s}} m^2\right)$ for direct collocation (Hespanhol and Quirynen, 2018), where N_{s} denotes the number of collocation intervals. However, the cost of solving the block-structured QP subproblems is typically $\mathcal{O}\left(N_{\mathrm{s}} m^3\right)$. For example, the sparsity exploiting `PRESAS` active-set QP solver (Quirynen et al., 2018b) enjoys a setup computational complexity of $\mathcal{O}\left(N_{\mathrm{s}} m^3\right)$ and a per iteration complexity of $\mathcal{O}\left(N_{\mathrm{s}} m^2\right)$.

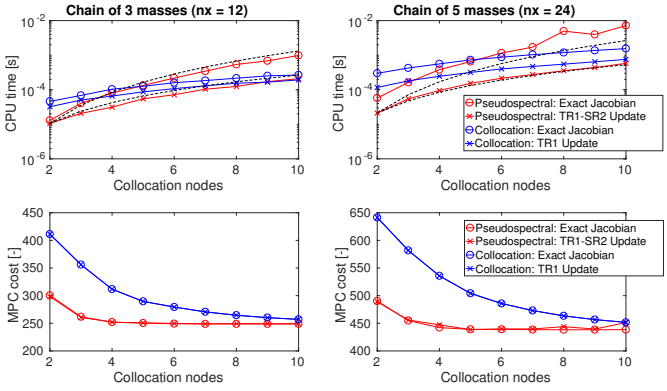

Fig. 1. Average computation time per RTI step and overall closed-loop cost of NMPC, for a varying number N_{s} of shooting intervals in direct collocation (with $N = 2$) or a varying number N of collocation nodes in the pseudospectral method, respectively.

On the other hand, a pseudospectral method converges exponentially to a smooth continuous time optimal control solution (Rao, 2010) for an increasing degree N of the collocation polynomial. Alternatively, a piecewise constant control parametrization is typically used in combination with direct multiple shooting (Bock and Plitt, 1984) or direct collocation (Biegler, 1984). Figure 1 shows the resulting trade-off between closed-loop control performance and computational cost, using an increasing number of collocation intervals N_{s} (direct collocation) or an increasing polynomial degree N (pseudospectral), for NMPC on the chain with $n_{\mathrm{m}} = 3$ or 5 masses. The results for direct collocation are based on a Gauss-Legendre (GL) method with $N = 2$ nodes for each interval. Figure 1 shows the performance for both the exact Jacobian and the TR1 based Newton-type optimization algorithms. Note that the black dashed lines illustrate, respectively, a computational cost that increases with a 2^{nd} or 3^{rd} order of complexity with the number N of collocation nodes.

Note that an alternative NMPC implementation could be based on a N-degree polynomial, for both the state and control parametrization, over each of the N_{s} collocation intervals in order to combine the advantages from both approaches for optimal control, such as in a spectral patching (Fahroo and Ross, 2000) or in a pseudospectral knotting method (Ross and Fahroo, 2004).

Table 1. Average timing results (in ms) of pseudospectral based NMPC for the chain of masses using $N = 8$ Gauss collocation nodes.

		Gauss-Newton with TR1	
$n_\mathrm{m} = 3, n_\mathrm{x} = 12$	Exact	Alg. 1 (TR1)	Alg. 2 (TR1-SR2)
Linearization	0.474	0.093	0.084
Dense QP solution	0.020	0.021	0.016
Total RTI step	**0.539**	**0.161**	**0.124**
$n_\mathrm{m} = 5, n_\mathrm{x} = 24$	Exact	Alg. 1 (TR1)	Alg. 2 (TR1-SR2)
Linearization	4.856	0.295	0.296
Dense QP solution	0.023	0.042	0.019
Total RTI step	**4.961**	**0.419**	**0.355**
$n_\mathrm{m} = 7, n_\mathrm{x} = 36$	Exact	Alg. 1 (TR1)	Alg. 2 (TR1-SR2)
Linearization	15.403	0.628	0.609
Dense QP solution	0.024	0.024	0.018
Total RTI step	**15.560**	**0.782**	**0.682**

5.2 Detailed Computational Performance

Table 1 shows the average computation times of the closed-loop NMPC simulation results using $N = 8$ Gauss collocation nodes for the chain of $n_\mathrm{m} = 3$, 5 and 7 masses. The table shows the detailed timing results for pseudospectral based NMPC, using either the exact Jacobian or the TR1 based Jacobian update scheme. It can be observed that the computation time for the problem linearization and condensing procedure can be reduced significantly based on the TR1 method, resulting in a speedup of about factor 4, 10 and 20, respectively, for the chain of $n_\mathrm{m} = 3$, 5 and 7 masses. On the other hand, the closed-loop NMPC performance is indistinguishable for the exact Jacobian and the TR1 based RTI scheme as shown earlier in Figure 1. Note that the additional speedup of using Algorithm 2 instead of 1 is small for this particular case study, given the small number of control inputs $n_\mathrm{u} = 3$ and therefore the relatively small dimension of the dense QP in (9), compared to the amount of state variables $n_\mathrm{x} = 6(n_\mathrm{m} - 1)$.

6. CONCLUSIONS

This paper proposed a lifted Newton-type optimization method for pseudospectral based nonlinear MPC (NMPC), using a rank-one Jacobian update formula in combination with numerical condensing and expansion of the collocation variables. We showed how the condensed Hessian can be updated directly, using either a symmetric rank-two or a rank-three update, in case that a quasi-Newton type method is used to approximate the Hessian of the Lagrangian. The proposed pseudospectral optimization algorithm has a quadratic computational complexity of $\mathcal{O}\left(N^2 m^2\right)$, compared to the typical complexity of $\mathcal{O}\left(N m^3\right)$ for sparsity exploiting algorithms based on direct collocation. A preliminary C code implementation has shown to allow considerable numerical speedups for the NMPC case study of the nonlinear chain of masses.

REFERENCES

Andersson, J. (2013). *A General-Purpose Software Framework for Dynamic Optimization*. Ph.D. thesis, Leuven.

Betts, J. (2010). *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*. SIAM.

Biegler, L.T. (1984). Solution of dynamic optimization problems by successive quadratic programming and orthogonal collocation. *Comp. Chem. Eng.*, 8, 243–248.

Bock, H.G. and Plitt, K.J. (1984). A multiple shooting algorithm for direct solution of optimal control problems. In *Proc. IFAC World Congr.*, 242–247. Pergamon Press.

Broyden, C.G. (1967). Quasi-Newton methods and their application to function minimization. *Math. Comp.*, 21.

Conn, A.R., Gould, N.I.M., and Toint, P.L. (1991). Convergence of quasi-newton matrices generated by the symmetric rank one update. *Math. Prog.*, 50, 177–195.

Diehl, M., Ferreau, H.J., and Haverbeke, N. (2009). Efficient numerical methods for nonlinear MPC and moving horizon estimation. In *Nonlinear model predictive control*, volume 384, 391–417. Springer.

Diehl, M., Findeisen, R., Allgöwer, F., Bock, H.G., and Schlöder, J.P. (2005). Nominal stability of the real-time iteration scheme for nonlinear model predictive control. *IEE Proc.-Control Theory Appl.*, 152(3), 296–308.

Diehl, M., Walther, A., Bock, H.G., and Kostina, E. (2010). An adjoint-based SQP algorithm with quasi-Newton Jacobian updates for inequality constrained optimization. *Opt. Meth. Softw.*, 25(4), 531–552.

Fahroo, F. and Ross, I.M. (2000). A spectral patching method for direct trajectory optimization. *Journal of the Astronautical Sciences*, 48(106), 269–286.

Griewank, A. (2000). *Evaluating Derivatives, Principles and Techniques of Algorithmic Differentiation*. Number 19 in Frontiers in Appl. Math. SIAM, Philadelphia.

Griewank, A. and Walther, A. (2002). On Constrained Optimization by Adjoint based quasi-Newton Methods. *Optimization Methods and Software*, 17, 869–889.

Hairer, E. and Wanner, G. (1991). *Solving Ordinary Differential Equations II – Stiff and Differential-Algebraic Problems*. Springer, Berlin Heidelberg, 2nd edition.

Hespanhol, P. and Quirynen, R. (2018). A real-time iteration scheme with quasi-Newton Jacobian updates for nonlinear model predictive control. In *Proc. ECC*.

Quirynen, R. and Diehl, M. (2018). Lifted Newton-type optimization for pseudospectral methods in nonlinear model predictive control. In *Am. Contr. Conf. (ACC)*.

Quirynen, R., Gros, S., and Diehl, M. (2018a). Inexact Newton-type optimization with iterated sensitivities. *SIAM Journal on Optimization*, 28(1), 74–95.

Quirynen, R., Knyazev, A., and Di Cairano, S. (2018b). Block structured preconditioning within an active-set method for real-time optimal control. In *Proc. ECC*.

Quirynen, R., Gros, S., Houska, B., and Diehl, M. (2017). Lifted collocation integrators for direct optimal control in ACADO toolkit. *Math. Progr. Comp.*, 9(4), 527–571.

Rao, A.V. (2010). A survey of numerical methods for optimal control. *Adv. Astron. Sciences*, 135(1).

Rawlings, J.B., Mayne, D.Q., and Diehl, M.M. (2017). *Model Predictive Control: Theory, Computation, and Design*. Nob Hill, 2nd edition.

Ross, I.M. and Fahroo, F. (2004). Pseudospectral knotting methods for solving nonsmooth optimal control problems. *J. Guidance, Control, Dynamics*, 27(3), 397–405.

Wirsching, L., Bock, H.G., and Diehl, M. (2006). Fast NMPC of a chain of masses connected by springs. In *Proc. IEEE Int. Conf. Contr. Appl.*, 591–596.