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Projection-Based Anti-Windup for Multivariable Control with Heat Pump Application

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***Abstract*— We present an anti-windup method in which the Hanus conditioning technique is combined with a user-designed projection of the reference onto the set of feasible steady-state points. This hybrid approach allows the designer to define a policy for steady-state reference tracking which is used to define the reference projection in the case that one or more control inputs saturate. The projection is computed only when the reference input changes, and is therefore less computationally taxing when compared to a command governor or model predictive control, for some applications. We demonstrate the method with an $\mathcal{H}_\infty$ loop-shaping controller for a multi-zone heat pump system in simulation.**

I. Introduction

In this paper we consider control of multivariable, linear time-invariant plants with limits on the control input. For such systems, there are two broad categories of control system design. On the one hand, we may consider the limits *explicitly* in the design, as in Model Predictive Control (MPC) [1], [2]. However, this may require the solution of a constrained optimization problem, and the computational requirements may exceed available resources. On the other hand, we may adopt a more *implicit* approach, and design a controller by first ignoring the hard limits and later modifying the resulting control law to account for them. The latter approach includes techniques such Command Governors (CG) [3] and also anti-windup [5]-[10] which must be done with careful consideration of the plant dynamics [11].

In this paper we adopt the second approach and address a problem that occurs for multivariable plants that incorporate anti-windup, namely loss of steady-state reference tracking due to activation of an input constraint. The problem is best illustrated by an example that motivates this work. Consider the four-zone heat pump shown in Fig. 1. We design an $\mathcal{H}_\infty$ loop-shaping feedback controller diagrammed in Fig. 2, with proportional-integrator (PI) weights $\tilde{\mathcal{W}}_1$ at the six inputs u (the compressor speed, CF, and the five Electronic Expansion Valve settings, EEVi, $1 = 1, \ldots, 4, m$) in order to meet disturbance rejection and reference tracking requirements, as described in [12]. In Fig. 2, $\mathcal{P}$ represents the plant, $\mathcal{W}_2$ is the output weight, $\mathcal{K}_s$ is the robustifying compensator, and $\mathcal{K}_0 = -\mathcal{K}_s(0)\mathcal{W}_2(0)$.

For this plant, there are six performance outputs in the vector y: the four room temperatures, $T_{R1}, \ldots, T_{R4}$, and two other temperatures that are in the refrigerant circuit, internal to the machine, and beyond our scope. Each of the inputs has a hard limit, so the PI weights are realized in Hanus form [6], which is indicated with the tilde above $\tilde{\mathcal{W}}_1$, to implement anti-windup. This design, which is described in more detail in Section IV, provides a good robustness margin, meets tracking, disturbance rejection, and transient response requirements, and also provides closed-loop stability in the event that any of the inputs saturate (which is not a natural consequence of the $\mathcal{H}_\infty$ design, but can be shown by analysis).

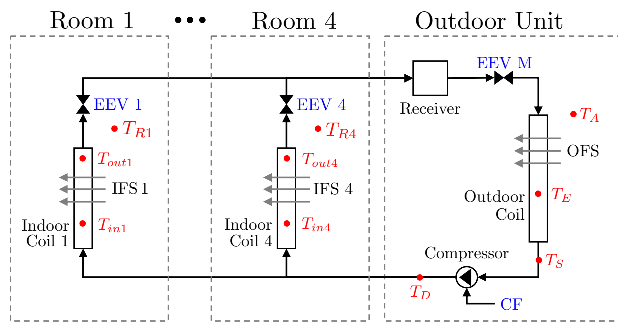

Fig. 1. Heat pump with four rooms, six control inputs (blue) and 16 measured temperature outputs (red).

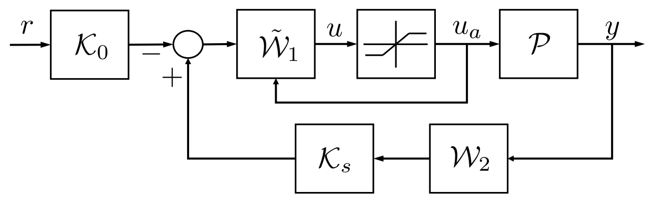

Fig. 2. Heat pump $\mathcal{P}$ with $\mathcal{H}_\infty$ loop-shaping controller in Hanus form.

However, when *any* of the inputs saturate, then *all* of the performance signals experience a non-zero steady-state tracking error. Fig. 3 shows the closed-loop system response to a 4°C step input in the Room 1 reference set-point. This causes three inputs to saturate, but we see that all four room temperatures (and the two other machine temperatures, not shown) have a non-zero steady-state tracking error. Yet, because three inputs remain unsaturated, we should be able to modify the controller and recover steady-state tracking for some of these performance variables. This is the subject of this paper.

This problem is naturally addressable with MPC or with a CG. These methods can enforce constraints in transient as well as steady-state, but can be computationally burdensome.

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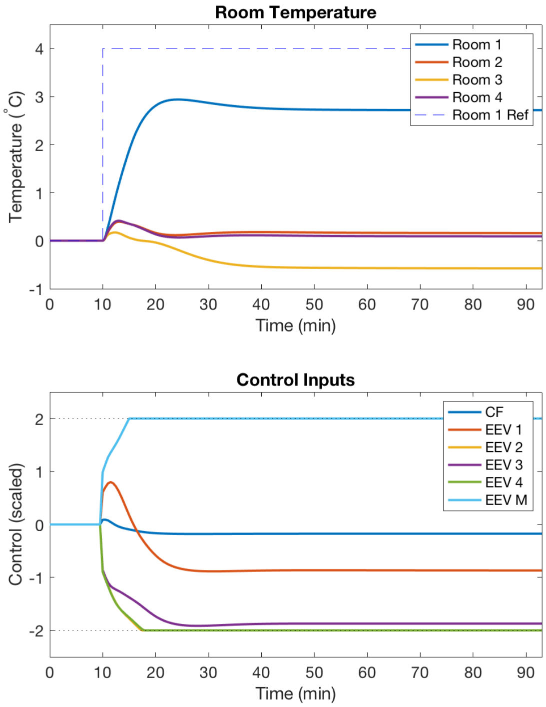

Fig. 3. Closed-loop heat pump response to a $4°\mathrm{C}$ step in Room 1 reference at $t = 10\,\mathrm{min}$, while keeping the other five references at zero. Because three control inputs saturate, all of the performance variables have non-zero steady-state error.

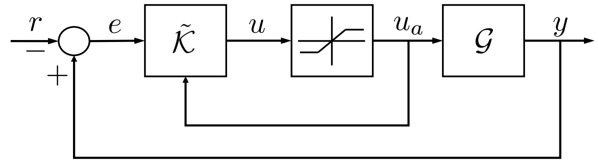

Fig. 4. General control-loop with actuator saturation in Hanus form.

Our interest here is also to enforce input constraints in both transient and steady-state, but to require tracking of a user-defined reference policy only in the steady-state. For example, one such policy is to maximize the number (cardinality) of references that have zero steady-state error. For our heat pump, this would result in the maximum number of room temperatures meeting their reference set-points in the steady-state, when input constraints are active.

Related work includes [13], [14], in which the Hanus conditioning technique is extended and a functional is introduced such that an *optimized* realizable reference is applied to the controlled plant. The cost functional can be designed to implement a reference tracking policy. The result is similar to this paper, but the optimization requires relatively expensive computations. Furthermore, no distinction is made between transient and steady-state reference tracking. Our requirement is only for steady-state tracking of a user-defined reference policy.

In this paper we propose to define a metric that represents a desired reference tracking policy, and to use the metric to compute a projection of the reference onto the set of steady-state feasible references. The projected reference is applied to the closed-loop system in which the Hanus conditioning technique is used for anti-windup and input constraint enforcement. This allows for a kind of separation of concerns, in which a nominal controller can be designed, neglecting the input saturation, and a metric is independently designed and used to project references onto the feasible set, in order to satisfy a desired policy for reference tracking in the event of control input saturation. This projection does not require any state estimation, and is computed only when the reference set-point changes, so it has a relatively low computational burden.

The paper is organized as follows. In Section II we define terms, review the Hanus anti-windup method, and define *output admissible sets* of references that do not lead to controller saturation in the steady-state. In Section III, we show that the Hanus method can be interpreted as a projection of the reference onto the output admissible set, but the metric that defines the projection is completely determined by parameters of the linear controller and cannot be separately *designed* to meet a steady-state tracking policy. We then present our main result, in which a feasible steady-state reference is first computed via a projection that is *designed* to satisfy a given steady-state tracking policy, and then the Hanus conditioning technique is used for anti-windup and input constraint enforcement. In Section IV, we return to the multi-zone heat pump example and show how the method improves reference tracking in the event of control input saturation.

Throughout this paper we use a positive feedback convention that is common in the $\mathcal{H}_\infty$ loop-shaping literature [15], [16]. We use the calligraphic font for matrix-valued transfer functions e.g., $\mathcal{P}$ denotes the plant transfer function and $\mathcal{P}(0)$ signifies the steady-state gain of $\mathcal{P}$, and we use the italics roman font to represent matrices, e.g., A. We also use calligraphic font to represent sets.

II. Preliminaries

Consider a strictly proper, stable linear time-invariant system $\mathcal{G}$ realized as

$$x(t+1) = Ax(t) + Bu_a(t),$$
$$y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u_a(t) \in \mathbb{R}^{n_u}$ is the actual control input, and $y(t) \in \mathbb{R}^{n_y}$ is the system output. We assume $\mathcal{G}$ is in feedback with compensator $\tilde{\mathcal{K}}$, as shown in Fig. 4, and that $\mathcal{G}$ is square, so that $n_u = n_y$. The main results are extended to non-square plants in Section IV. Note that the feedback loop in Fig. 2 is an instance of the feedback loop shown in Fig. 4, with $\tilde{\mathcal{K}} = \tilde{\mathcal{W}}_1$ and $\mathcal{G} = \mathcal{K}_s\mathcal{W}_2\mathcal{P}$, neglecting the constant input gain $\mathcal{K}_0$.

The controller output $u(t)$ is subject to saturation such that the saturated input $u_a(t)$ enters the plant, and the set of admissible control inputs is

$$\mathcal{U} := \left\{ u \in \mathbb{R}^{n_u} : u_{i,\min} \leq u_i \leq u_{i,\max},\, 1 \leq i \leq n_u \right\}. \quad (1)$$

In Fig. 4 we assume that a compensator $\mathcal{K}$ is designed to meet a set of linear performance requirements for the feedback loop, neglecting the input saturation, and that it includes integral action in its diagonal elements. In the simplest case it is a diagonal system of Proportional Integral (PI) compensators. To account for input saturation, it is realized in so-called *Hanus form* as $\tilde{\mathcal{K}}$, taking as input the tracking error e as well as the output of the saturation block u_a, as described in the next section.

A. Hanus Conditioning Technique

In case of actuator saturation, integration in $\mathcal{K}$ must be stopped to prevent windup. This can be achieved by using the Hanus conditioning technique [6], if $\mathcal{K}$ is invertible and minimum phase, which is the case when $\mathcal{K}$ is a diagonal system of PI compensators, for example.

Let $\mathcal{K}$ be realized as

$$\mathcal{K} \overset{S}{=} \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right], \quad (2)$$

where $\det D_K \neq 0$. Then the Hanus conditioning technique is realized as

$$x_K(t+1) = A_K x_K(t) + B_K(\tilde{r}(t) + y(t)), \quad (3)$$
$$u(t) = C_K x_K(t) + D_K(r(t) + y(t)), \quad (4)$$
$$u_a(t) = \mathrm{sat}(u(t)), \quad (5)$$
$$\tilde{r}(t) = r(t) + D_K^{-1}\left(u_a(t) - u(t)\right), \quad (6)$$

where $x_K(t) \in \mathbb{R}^{n_{x_K}}$ is the state, $r(t) \in \mathbb{R}^{n_y}$ is the reference, and $\tilde{r}(t) \in \mathbb{R}^{n_y}$ is the so-called *realizable reference*, that, if applied instead of $r(t)$ would have lead to $u_a(t)$ being equal to $u(t)$. In implementation, the *self-conditioned*, or *Hanus* form of the controller is often used. The Hanus form is obtained by substituting (6) into (3), expressed as

$$\tilde{\mathcal{K}} \overset{S}{=} \left[\begin{array}{c|cc} A_K - B_K D_K^{-1} C_K & 0 & B_K D_K^{-1} \\ \hline C_K & D_K & 0 \end{array} \right], \quad (7)$$

where the input vector is given by a vertical concatenation of $e(t) = y(t) - r(t)$ and $u_a(t)$ in that order.

For the box constraints on u, the saturation function $\mathrm{sat}(\cdot)$ can be defined as orthogonal projection onto $\mathcal{U}$, i. e.,

$$\mathrm{sat}(u(t)) := \pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u(t)), \quad (8)$$

where

$$\pi_{\mathcal{S}}^{d}(a) := \arg\min_{b \in \mathcal{S}} d(a,b),$$

and where $S \subset \mathbb{R}^{n_a}$, $a \in \mathbb{R}^{n_a}$, and d is a metric on $\mathbb{R}^{n_a}$. We denote the metric that is induced by the Euclidian norm $\|\cdot\|$ by $d_{\|\cdot\|}$.

B. Admissible Sets

Handling actuator saturation can be interpreted as an input constraint adherence problem. Therefore, we now use the notion of admissible sets to be able to describe the benefits and shortcomings of different approaches that handle actuator saturation.

Let an autonomous system be given by $\xi(t+1) = g(\xi(t))$, where $g : \mathbb{R}^n \to \mathbb{R}^n$, and let $\mathcal{X} \subseteq \mathbb{R}^n$ be the set of admissible states. Then we can define the sequence of sets

$$\mathcal{O}_k := \{\xi \in \mathbb{R}^n : \xi(0) = \xi,\, \xi(t+1) = g(\xi(t)),$$
$$\xi(t) \in \mathcal{X}, \text{ for all } t = 0, 1, \ldots, k\},$$

which denote the sets of states that will stay admissible for at least k time steps. Its limit, $\mathcal{O}_\infty$, is called the maximal positive invariant set, i. e. the set of all states, that stay admissible under g for all times. These sets are often used in MPC (cf. [2, Chapter 11]), and CG ([3]).

For this work, we introduce another sequence of sets, defined by

$$\mathcal{A}_k := \{\xi \in \mathbb{R}^n : \xi(0) = \xi,\, \xi(t+1) = g(\xi(t)),$$
$$\text{for all } t = 0, 1, \ldots, k-1,\, \xi(k) \in \mathcal{X}, \},$$

which denotes the set of states that are admissible after k steps. We call its limit, $\mathcal{A}_\infty$, the set of *eventually admissible* states.

Referring back to Fig. 4, let the discrete-time closed-loop system from the reference r to the control signal u with transfer function

$$\mathcal{P}_{cl} := \mathcal{K}(I - \mathcal{G}\mathcal{K})^{-1} \quad (9)$$

have the realization

$$\mathcal{P}_{cl} \overset{S}{=} \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array} \right].$$

Then we can define the augmented autonomous system for $\xi = [x^\mathsf{T}\, r^\mathsf{T}]^\mathsf{T}$ as

$$\xi(t+1) = g(\xi(t)) = \underbrace{\begin{bmatrix} A_{cl} & B_{cl} \\ 0 & I_{n_y} \end{bmatrix}}_{A_{\mathrm{ex}}} \xi(t), \quad (10)$$

where $I_{n_y} \in \mathbb{R}^{n_r \times n_r}$ denotes the identity matrix, to investigate the system behavior for constant references [3].

Then, the admissible set is given by

$$\mathcal{X} = \left\{ [x^\mathsf{T}\, r^\mathsf{T}]^\mathsf{T} : C_{cl} x + D_{cl} r \in \mathcal{U} \right\}. \quad (11)$$

Note that by the definition of g in (10), the reference is constant, so $\mathcal{O}_\infty$ is the set of states and references that, if the references are constant, will not lead to a constraint violation.

Next, we use the structure of ξ to define admissible references depending on either a measured or an estimated state of the closed-loop system. In this way, we define

$$\mathcal{R}_{\mathcal{O}_k}(x) := \left\{ r : [x^\mathsf{T}\, r^\mathsf{T}]^\mathsf{T} \in \mathcal{O}_k \right\}, \text{ and}$$
$$\mathcal{R}_{\mathcal{A}_k}(x) := \left\{ r : [x^\mathsf{T}\, r^\mathsf{T}]^\mathsf{T} \in \mathcal{A}_k \right\}.$$

Note that we usually omit the dependency of $\mathcal{R}_{\mathcal{O}_k}$ $\mathcal{R}_{\mathcal{A}_k}$ on x. These, and the previously defined sets, can be expressed algebraically with sets of inequalities. The set $\mathcal{X}$, as defined in (11), is given by all ξ that satisfy

$$\underbrace{\begin{bmatrix} C_{cl} & D_{cl} \\ -C_{cl} & -D_{cl} \end{bmatrix}}_{H} \xi \leq \begin{bmatrix} u_{1,\max} \\ \vdots \\ u_{n_u,\max} \\ -u_{1,\min} \\ \vdots \\ -u_{n_u,\min} \end{bmatrix} =: b,$$

where $\leq$ acts componentwise, while $\mathcal{O}_k$ for the linear system defined in (10) is a polyhedron [3], which can be expressed as

$$\begin{bmatrix} H \\ HA_{\text{ex}} \\ \vdots \\ HA_{\text{ex}}^k \end{bmatrix} \xi \leq \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix}. \tag{12}$$

Similarly, $\mathcal{A}_k$ can be expressed

$$HA_{\text{ex}}^k \xi \leq b.$$

For algebraic representations of $\mathcal{R}_{\mathcal{A}_k}$ and $\mathcal{R}_{\mathcal{O}_k}$, the inequalities describing $\mathcal{A}_k$ and $\mathcal{O}_k$ can simply be rearranged such that all components depending on x are on the right-hand side such that we get for $\mathcal{R}_{\mathcal{O}_k}$

$$\begin{bmatrix} H \\ HA_{\text{ex}} \\ \vdots \\ HA_{\text{ex}}^k \end{bmatrix} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} r \leq \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix} - \begin{bmatrix} H \\ HA_{\text{ex}} \\ \vdots \\ HA_{\text{ex}}^k \end{bmatrix} \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} x,$$

and for $\mathcal{R}_{\mathcal{A}_k}$

$$HA_{\text{ex}}^k \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} r \leq b - HA_{\text{ex}}^k \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} x.$$

Remark 1: Note that $\mathcal{R}_{\mathcal{A}_\infty}$, which is given by

$$\left(\sum_{j=0}^{\infty} C_{cl} A^j B_{cl} + D_{cl} \right) \begin{bmatrix} I \\ -I \end{bmatrix} r \leq b,$$

is independent of the state x, since A_{cl} is exponentially stable and therefore

$$\lim_{k\to\infty} HA_{\text{ex}}^k = \begin{bmatrix} I \\ -I \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} \lim_{k\to\infty} \begin{bmatrix} A_{cl}^k & \sum_{j=0}^{k} A_{cl}^j B_{cl} \\ 0 & I_{n_y} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \sum_{j=0}^{\infty} C_{cl} A_{cl}^j B + D_{cl} \\ 0 & -\sum_{j=0}^{\infty} C_{cl} A_{cl}^j B - D_{cl} \end{bmatrix}.$$

For that reason we have that $\mathcal{R}_{\mathcal{A}_\infty}$ is equivalent to the *set of feasible steady state points* of the stable square system G, as defined in [17], since the integral action ensures zero steady-state tracking error. This set of feasible steady state points is defined by all references r that can be expressed by

$$r = C \sum_{k=0}^{\infty} A^k B \bar{u} = C\,(I - A)^{-1} B \bar{u},$$

with $\bar{u} \in \mathcal{U}$. Since the DC-gain of $\mathcal{GK}(I - \mathcal{GK})^{-1}$ is the identity matrix if we have zero steady-state tracking error, we have that the DC-gain of $\mathcal{G}$ is the inverse of the DC-gain of $\mathcal{K}(I - \mathcal{GK})^{-1}$.

C. Command Governors

A Command Governor (CG) is used to enforce constraints by adjusting the reference depending on a measured or an estimated state of the closed-loop system. Typically, a projection of a user defined reference r_{user} onto the set $\mathcal{R}_{\mathcal{O}_\infty}$ is computed and applied to the plant with a metric d that can be specified by the user. To compute $\pi^d_{\mathcal{R}_{\mathcal{O}_\infty}}(r_{\text{user}})$, a constrained optimization problem is solved numerically, and the closed-loop system state must be estimated. This may be more demanding than the other computations required to determine the control signal. Because of this, $\pi_{\mathcal{R}_{\mathcal{O}_\infty}}(r_{\text{user}})$ is usually not computed at every time step. A comprehensive literature review on reference and command governors is given in [3].

III. Main Results

We first show that the Hanus conditioning technique can be interpreted as a projection onto the admissible set $\mathcal{R}_{\mathcal{O}_0}$ of $\mathcal{P}_{cl}$ subject to $\mathcal{U}$, with a metric that depends only on the matrix D_K, and therefore cannot be designed independently of $\mathcal{K}$. In what follows, $\|v\|_Q = \sqrt{v^T Q v}$ for vector v and positive definite matrix Q.

Theorem 1: Let $\mathcal{G}$ be strictly proper. Then the realizable reference $\tilde{r}$ as defined in (6) is the minimizer for the optimization problem

$$\tilde{r} = \arg\min_{p \in \mathcal{R}_{\mathcal{O}_0}} \|r - p\|^2_{D_K^\intercal D_K}.$$

Proof: First, we derive the admissible set $\mathcal{R}_{\mathcal{O}_0}$. For that, we compute the state space form of $\mathcal{K}(I - \mathcal{GK})^{-1}$. Therefore, let the state space representation of $\mathcal{GK}$ be given by

$$\mathcal{GK} \stackrel{S}{=}: \left[\begin{array}{c|c} A_l & B_l \\ \hline C_l & D_l \end{array}\right].$$

Since $\mathcal{G}$ is strictly proper, we have that $D_l = 0$. Then $(I - \mathcal{GK})^{-1}$ is given in state space form by

$$(I - \mathcal{GK})^{-1} \stackrel{S}{=}: \left[\begin{array}{c|c} -A_l + B_l C_l & -B_l \\ \hline -C_l & I \end{array}\right].$$

Finally, $\mathcal{K}(I - \mathcal{GK})^{-1}$ is given by

$$\left[\begin{array}{cc|c} A_K & B_K C_l & B_K \\ 0 & -A_l + B_l C_l & -B_l \\ \hline C_K & -D_K C_l & D_K \end{array}\right].$$

Thus, we have

$$\mathcal{R}_{\mathcal{O}_0}(x) = \{r : D_K r + F x \in \mathcal{U}\}, \tag{13}$$

where $F = \begin{bmatrix} C_K & -D_K C_l \end{bmatrix}$.

Next, we show $\tilde{r} \in \mathcal{R}_{\mathcal{O}_0}$. We have that $u = Fx + D_K r$ and define $\tilde{u} := \pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u) - u$. Therefore, we have

$$\begin{aligned} D_K \tilde{r} + Fx &= D_K r + \tilde{u} + Fx \\ &= u + \tilde{u} = \pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u), \end{aligned}$$

where $\pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u) \in \mathcal{U}$ by the definition of the projection operator $\pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u)$. Thus, $\tilde{r} \in \mathcal{R}_{\mathcal{O}_0}$ by definition (13) of the set $\mathcal{R}_{\mathcal{O}_0}$.

It remains to show that $\tilde{r}$ is optimal. Let $u(p) = Fx + D_K p$ be the control input produced by a reference value p. Then

$$\begin{aligned} \|u - u(p)\|^2 &= \|Fx + D_K r - Fx - D_K p\|^2 \\ &= \|D_K(r-p)\|^2 \\ &= \|r - p\|^2_{D_K^\intercal D_K}. \end{aligned}$$

Since $\tilde{r} = r - D_K^{-1}\left(\pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u) - u\right)$ we have

$$\begin{aligned} \|r - \tilde{r}\|_{D_K^\intercal D_K} &= \left\| D_K^{-1}\left(\pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u) - u\right)\right\|^2_{D_K^\intercal D_K} \\ &= \left\| u - \pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u)\right\|^2. \end{aligned}$$

By definition of the projection operator $\pi_{\mathcal{U}}^{d_{\|\cdot\|}}(u)$ this means

$$\begin{aligned} \|r - \tilde{r}\|^2_{D_K^\intercal D_K} &\leq \|u - u(p)\|^2 \\ &= \|r - p\|^2_{D_K^\intercal D_K} \end{aligned}$$

for all p such that $Fx + D_K p \in \mathcal{U}$. In other words,

$$\|r - \tilde{r}\|^2_{D_K^\intercal D_K} \leq \|r - p\|^2_{D_K^\intercal D_K},$$

for all $p \in \mathcal{R}_{\mathcal{O}_0}$. ∎

This result states that the Hanus conditioning technique is in fact a computationally efficient projection of r onto the set of admissible references. However, the metric that defines the projection cannot be designed independent of $\mathcal{K}$ to realize a desired reference tracking policy. Next, the steady-state behavior of the Hanus conditioning technique is studied.

Assumption 1: Let $\mathcal{G}$, $\tilde{\mathcal{K}}$, and the unconstrained closed-loop system be asymptotically stable. Furthermore, let $\mathcal{G}$ be strictly proper.

Assumption 2: Let B_K, C_K, and D_K be invertible and A_K be the identity matrix.

These assumptions may seem very limiting, but in practice only part of the actual controller, e.g., the weight $\mathcal{W}_1$ in Fig. 2, must satisfy these Assumptions because the rest of the controller dynamics can be lumped into the plant $\mathcal{G}$. This is a common situation in, for example, an $\mathcal{H}_\infty$ loop-shaping controller design.

Theorem 2: If the system satisfies Assumptions 1 and 2, then the realizable reference $\tilde{r}$ converges to a given r if and only if r is a feasible steady-state point.

Proof: Since Assumption 1 is satisfied, by the theorem in [6] the constrained closed-loop is asymptotically stable. Therefore, the signals $u(t), y(t)$, and $u_a(t)$ converge to finite values. From $\tilde{r}(t) = r(t) + D_K^{-1}(u_a(t) - u(t))$ we have that $\tilde{r}(t) = r(t)$ is equivalent to $u(t) = u_a(t)$. By definition of $u_a(t)$ we have that $u_a(t) \in \mathcal{U}$ for all $t \in \mathbb{Z}^+$. We denote the limit of $u_a(t)$ for $t \to \infty$ by u_a^∞. In this way, we have that

$$\lim_{t\to\infty} y(t) = C(I - A)^{-1} B u_a^\infty,$$

and

$$\lim_{t\to\infty} x_K(t) = (I - A_K - B_K D_K^{-1} C_K)^{-1} B_K D_K^{-1} u_a^\infty.$$

Therefore, we have

$$\begin{aligned} \lim_{t\to\infty} u(t) = C_K(I - A_K - B_K D_K^{-1} C_K)^{-1} B_K D_K^{-1} u_a^\infty \\ + D_K(r - C(I-A)^{-1} B u_a^\infty) = u_a^\infty. \end{aligned}$$

Since $A_K = I$, and $B_K D_K^{-1} C_K$ is invertible, this can be rearranged such that

$$C_K(I - A_K - B_K D_K^{-1} C_K)^{-1} B_K D_K^{-1} = I,$$

we have that

$$D_K(r - C(I-A)^{-1} B u_a^\infty) = 0 \Leftrightarrow r = C(I-A)^{-1} B u_a^\infty.$$

∎

Based on the results of Theorems 1 and 2, we propose to combine the reference projection associated with a Command Governor with the Hanus form of anti-windup, as shown in Fig. 5. The reference set-point r is first projected to a point on $\mathcal{R}_{\mathcal{A}_\infty}$, which is optimal in terms of a metric d. We denote that projection by $\pi_\infty := \pi^d_{\mathcal{R}_{\mathcal{A}_\infty}}$. The output of the π_∞ block is a feasible steady-state point. The projected reference is applied to the system. Then, by Theorem 2, the closed-loop system will converge to that point in the steady-state. Because we require policy adoption only in the steady-state, which is sufficient for our specific use-case, we reduce the computational complexity compared to a conventional CG, because the optimization problem that computes the projection must be solved only when the reference set-point changes. This is advantageous compared to the methods described in [13], [14], in which the projection onto the set of realizable references is posed as a nonlinear programming problem that must be solved at each time step. The separation of optimization and constraint enforcement means we have the simplicity of the basic *Hanus* form while being able to use the flexibility of more advanced methods based on convex optimization.

Compared to MPC or CG, the computational burden is reduced for two reasons. First, the number of optimization problems is reduced from once per time step to once per reference update. Second, the number of constraints is reduced significantly. This can be seen in equation (12), in which the number of constraints describing $\mathcal{O}_k$ is given by $2kn_u$, whereas the number of constraints that describe $\mathcal{A}_\infty$ is limited to $2n_u$. Even though redundant constraints of $\mathcal{O}_k$ can be eliminated, which reduces the number of inequalities that describe $\mathcal{O}_k$, the number of constraints that have to be considered during optimization is significantly higher even with moderately chosen k for $\mathcal{O}_k$ compared to $\mathcal{A}_\infty$.

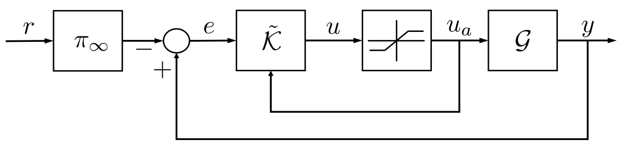

Fig. 5. Control loop with actuator saturation in Hanus form and projection onto set of feasible steady-state points.

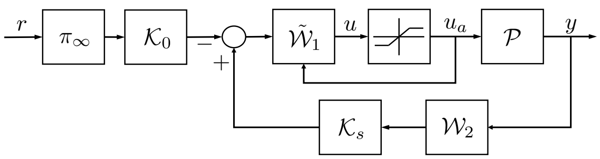

Fig. 6. $\mathcal{H}_\infty$ setup in Hanus form with projection onto feasible steady-state points.

IV. Application to $\mathcal{H}_\infty$ loop-shaping

In this section we return to the multi-zone heat pump example that was introduced in Section I. For the heat-pump, a detailed nonlinear system model was linearized and reduced resulting in a low-order, linear, discrete-time model $\mathcal{P}$. For details see [12]. $\mathcal{H}_\infty$ loop-shaping was used to design the feedback controller shown in Fig. 6, following the design process described in [18, Chapter 9.4], in which the frequency response $\mathcal{P}(j\omega)$ is shaped using the pre- and post-compensators $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. The weight $\mathcal{W}_1$ is a diagonal system of proportional-integral (PI) compensators, which are realized in Hanus form $\tilde{\mathcal{W}}_1$ for anti-windup, while $\mathcal{W}_2$ is a diagonal matrix of constants. Then the compensator $\mathcal{K}_s$ is computed by solving a robust stabilization problem that maximizes robustness with respect to normalized coprime factor uncertainty, which is a convenient and practical way to represent model uncertainty in vapor compression systems.

Note that Fig. 6 is an instance of Fig. 5 with $\mathcal{G} = \mathcal{K}_s\mathcal{W}_2\mathcal{P}$ and $\tilde{\mathcal{K}} = \tilde{\mathcal{W}}_1$. Also note that it is not necessary that $\mathcal{P}$ be square. In fact, the heat pump model $\mathcal{P}$ has 16 measured outputs, six control inputs, and four unmeasured disturbance inputs. Of the 16 measurements, six are chosen for regulation: the four room temperatures T_{Ri}, $1 \leq i \leq 4$, and other two internal machine temperatures that are beyond our scope. These are denoted "regulated outputs." Thus, $\mathcal{W}_2$ is 16×16, and $\mathcal{K}_s$ is 16×6. To prevent derivative kick [18], the reference is subtracted from the output of $\mathcal{K}_s$ and not from the output of $\mathcal{P}$. Therefore, in addition to the projection onto the set of feasible steady state points, a constant square prefilter $\mathcal{K}_0$ is applied to the references. It is given by the steady state gain of $-\mathcal{K}_s\mathcal{W}_2$ to get steady-state gain matching. But because $\mathcal{P}$ is not square, $\mathcal{K}_0$ is constructed from the columns of the steady-state gain of $\mathcal{K}_s\mathcal{W}_2$ that correspond to the regulated outputs.

Next we discuss the policy for reference tracking when inputs are saturated. Our objective is to minimize the number of room temperatures that do not match the given reference. The cost function associated with the minimization of the number of violated zone temperatures can be given as

$$J(r) = \|r - r_{\text{user}}\|_0,$$

where $\|\cdot\|_0$, denotes the zero pseudo norm, i.e., the Hamming distance from 0, which is defined by

$$\|v\|_0 := \operatorname{card}\{v_i : v_i \neq 0\},$$

where card denotes the set cardinality.

Since the optimization of the zero pseudo norm is known to be a computationally difficult problem [19], surrogate norms that lead to similar optimization results are often used instead. In [20], [21], [22] it is shown that optimizing the one norm can lead to similar results as a zero pseudo norm optimization. One norm optimization can be solved using linear programming, which can be efficiently and reliably solved with existing software such as qpOASES [23].

Thus for implementation on the heat pump, which has very limited computational resources, we solve the constrained optimization problem

$$\begin{cases} \text{minimize } \|r - r_\infty\|_1 \\ \text{subject to } r_\infty \in \mathcal{R}_{\mathcal{A}_\infty}, \end{cases}$$

for r_∞. The set $\mathcal{R}_{\mathcal{A}_\infty}$ is computed as described in Section II-B, where $\mathcal{K}$ and $\mathcal{G}$ in P_{cl} as defined in (9) are given by $\mathcal{K} := \mathcal{W}_1$ and $\mathcal{G} := \mathcal{K}_s\mathcal{W}_2\mathcal{P}$. The output r_∞, is then applied to the plant. Note that although we solve a linear programming problem, the calculation of r_∞ depends only on r, so it must be done only when the reference r changes. For the heat pump application, this is relatively rare, and the calculation need not complete within one clock cycle. This is in contrast to a CG, which solves one optimization per control cycle.

Fig. 7 plots the same step response as Fig. 3, but when the projection π_∞ is included as shown in Fig. 6. Using the reference projection, it is possible to enforce the constraints and implement the policy that in case of actuator saturation, the maximum number of room temperatures should be tracked with zero steady-state error. In this case, we track room temperature set-point in three of four rooms. Of course, as a consequence the steady-state tracking error in Room 1 is larger. It is interesting to note that only one constraint is active in the steady-state, in contrast to the conventional Hanus anti-windup shown in Fig. 3.

V. Conclusions

In this paper we have described a computationally simple method to modify reference signals for multi-variable linear time-invariant systems with input constraints. The method implements a designed policy for reference tracking in the steady-state when one or more inputs are saturated. This addresses a problem with conventional Hanus-type anti-windup for multivariable linear systems, namely that the closed-loop system will not achieve steady-state reference tracking when any one input saturates. The method is similar to the Command Governor, but with a lower computational burden because the modified reference needs to be computed

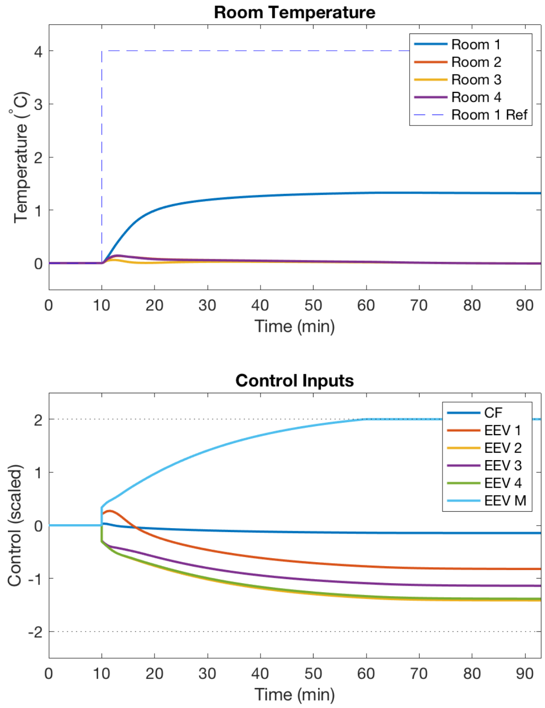

Fig. 7. Room temperature response (top) and scaled actuator response (bottom) to a 4°C step in the Room 1 reference, holding the other five references constant, with the projected reference and Hanus anti-windup. The reference policy is to maximize the number of zones that maintain zero steady-state tracking error. We see that one control input saturates, and that only the Room 1 temperature experiences a steady-state error, while the other three have zero steady-state tracking error.

only when the reference changes. Our method separates the concern of controller design for the nominal system and design and realization of a policy for tracking when constraints are active. We demonstrate the method on a multi-zone heat pump system with an $\mathcal{H}_\infty$ loop-shaping controller.

This can be extended in various ways. Disturbance rejection and enforcement of state constraints need to be investigated. This may require state estimation in the general case. Investigation of other metrics is also important, as are the relationships with command governors and model predictive control. Furthermore, the combination of projection with more advanced anti-windup methods such as [24] should be studied because these allow the application of hybrid approach to open-loop unstable plants.

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