MITSUBISHI ELECTRIC RESEARCH LABORATORIES
https://www.merl.com

On the Stability Properties of Perception-aware Chance-constrained MPC in Uncertain Environments

Bonzanini, Angelo Domenico; Mesbah, Ali; Di Cairano, Stefano

TR2021-145 January 11, 2022

Abstract

Perception systems for acquiring environment information and control systems for commanding a system operating in the environment are usually separately designed, but their performance is often interdependent: control decisions affect perception, e.g., the distance from an object affects its sensing, and perception affects control decisions, e.g., ar- eas may be traversed or avoided based on the amount of available information. Perception-aware control accounts for such an interdependence to optimize the overall performance. We recently proposed a perception-aware chance constrained MPC (PAC-MPC) that considers the impact of control on the evolution of the environment uncertainty, then used within chance constraints. In this paper we obtain a stabilizing design the PAC-MPC, by first determining stability conditions in a general nonlinear setting, and then deriving specify design rules for the linear-Gaussian case, which results in a specific choice of the cost function parameters and in design conditions for the estimation algorithms that determine uncertainty propagation. The results are illustrated by means of a numerical example.

IEEE Conference on Decision and Control (CDC) 2021

Mitsubishi Electric Research Laboratories, Inc.
201 Broadway, Cambridge, Massachusetts 02139

On the Stability Properties of Perception-aware Chance-constrained MPC in Uncertain Environments

Angelo D. Bonzanini, Ali Mesbah, and Stefano Di Cairano

***Abstract*— Perception-aware control systematically considers the interdependence between perception and control to optimize the overall performance of the closed-loop system subject to state and input-dependent uncertainty. That is, it accounts for the impact of control on sensing and of sensing on control. Recently, we proposed a perception-aware chance-constrained MPC (PAC-MPC) that considers the impact of control on the evolution of the environment uncertainty. In this paper, we obtain a stabilizing design for the PAC-MPC by first determining stability conditions in a general nonlinear setting and then deriving specific design rules for the linear-Gaussian case. The latter case results in a specific choice of the MPC cost function parameters, and in design conditions for that the estimation algorithm, that determine uncertainty propagation in the MPC prediction model, must satisfy.**

I. Introduction

In applications such as automotive, aerospace and robotics [1], [2], a system with *known* dynamics operates in an environment that is not precisely known, but only observed through measurements. In such cases, the design of the control system encompasses both the perception and control algorithms. While these algorithms are often independently designed, they are actually interdependent. Perception affects control, since the knowledge of the environment and its uncertainty may affect the control decisions. At the same time, control affects perception, since the acquisition of measurements and their quality often depends on system operation, e.g., where the sensors are pointed, how much processing is done on the measurements, or where the attention is focused. A case that is often of practical interest is when control actions affect the amount of uncertainty in the measurements, while the measured values are not directly affected. In statistical terms, this means that the covariance of the measurement is affected by the control actions - either directly through some decision variables, or indirectly based on the system state - while the mean remains unaffected.

Some previous works have proposed designs that account for the interaction between control and perception through the constraints [3], [4]. For the case where the environment measurement uncertainty depends on the system states and inputs, we proposed a perception-aware chance-constrained MPC (PAC-MPC) [5], which accounts for the effect of the uncertainty in the environment through the chance constraints [6], [7], while also considering the impact of the control actions on the environment uncertainty. Thus, reducing the environment uncertainty and achieving the control objective are coupled, enabling a systematic trade-off between perception and control.

Our work in [5] focused on formulating the PAC-MPC algorithm for the case of linear dynamics under several design assumptions. This paper extends our work in [5] by (i) considering general nonlinear dynamics for the system and the environment; and (ii) deriving the conditions for the PAC-MPC that ensure the stability of not only the system state, but also the uncertainty of the environment estimate. Moreover, in the case of linear system dynamics, we also show constructive conditions for designing the PAC-MPC to achieve the desired properties. As such, the PAC-MPC aims at stabilizing the system state and the uncertainty of the environment estimate, while the convergence of the mean of the estimate is assumed to be guaranteed by a proper design of the estimation algorithm. Although PAC-MPC shares some similarities with dual-objective MPC (e.g., see [8], [9]), it allows for simpler stability conditions because the uncertainty does not affect the system state evolution.

Section II, we summarize the models of system, environment, constraints and estimator, and describe PAC-MPC for the nonlinear case. In Section III, we investigate conditions for recursive feasibility in probability, and show how to satisfy them in the linear case. In Section IV, we investigate conditions for stability and we show how to satisfy them for the linear case. Section V demonstrates the results on an illustrative example, and Section VI the conclusions. Due to limited space, we only describe the main steps of the proofs.

Notation: $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_{+}$, $\mathbb{Z}$, $\mathbb{Z}_{0+}$, and $\mathbb{Z}_{+}$ are the sets of real, nonnegative real, positive real, integer, nonnegative integer, and positive integer numbers, respectively. Intervals are denoted by $\mathbb{Z}_{[a,b)} = \{z \in \mathbb{Z} : a \leq z < b\}$. For vectors x y, $[x]_i$ is i-th component, $(x, y) = [x^\top \ y^\top]^\top$, and for a positive (semi) definite matrix $Q > 0$, $(Q \geq 0)$, $\|x\|_Q^2 = x^\top Q x$. For a matrix A, $\|A\|_F$ is the Frobenius norm and tr the trace. $\mathbb{P}[A]$ is the probability of event A. Given the first and second central moments of a random variable, i.e., mean μ and covariance Σ, $\mathcal{M} = (\mu, \Sigma)$, for shortness, and $x \sim \mathcal{N}(\mu, \Sigma)$ indicates that x is normally distributed. A function $\alpha : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, $\alpha(0) = 0$, and of class $\mathcal{K}_\infty$ if also $\lim_{c\to\infty} \alpha(c) = \infty$. For compactness we write $g(z, x^{(a,b)}, y) = g(z, x^a, x^b, y)$.

II. PAC-MPC in Uncertain Environments

This section introduces the PAC-MPC for controlling a system subject to uncertain chance constraints, where the

A. D. Bonzanini and A. Mesbah are with the Department of Chemical and Biomolecular Engineering at the University of California, Berkeley, CA 94720, USA. {adbonzanini,mesbah}@berkeley.edu

S. Di Cairano is with Mitsubishi Electric Research Laboratories (MERL), Cambridge, MA. dicairano@ieee.org

environment uncertainty is dependent on the sensing.

A. Models of Known System and Uncertain Environment

We consider a *known* discrete-time system

$$x^s_{k+1} = f^s(x^s_k, u^s_k), \tag{1a}$$
$$y^s_k = h^s(x^s_k, u^s_k), \tag{1b}$$

where $x^s \in \mathbb{R}^{n_x}, u^s \in \mathbb{R}^{n_u}$, and $y^s \in \mathbb{R}^{n_y}$ are the *system* state, input, and output vectors, respectively. System (1) is subject to hard constraints on states and inputs, expressed as

$$x^s \in \mathcal{X}, \quad u^s \in \mathcal{U}, \tag{2}$$

where $\mathcal{X}$ and $\mathcal{U}$ are the admissible state and the input sets.

While (1) and (2) are assumed to be perfectly known, the system operates in an uncertain environment. The environment state dynamics and measurements are modeled as

$$x^e_{k+1} = f^e(x^e_k, \psi_k), \tag{3a}$$
$$y^e_k = q^e(x^e_k, \zeta_k, x^s_k, u^s_k), \tag{3b}$$

where $x^e \in \mathbb{R}^{m_x}$ and $y^e \in \mathbb{R}^{m_y}$ are the *environment* state and measurement, and ψ, ζ are random variables for the process noise and measurement noise with $\mathcal{M}^\psi = (\mu^\psi, \Sigma^\psi)$, $\mathcal{M}^\zeta = (\mu^\zeta, \Sigma^\zeta)$, respectively.

The environment (3) imposes additional constraints on system (1). Due to the uncertainty in (3), the constraints are modeled as linear individual chance constraints (ICCs)

$$\mathbb{P}\left[h^s_i x^s + h^e_i x^e \le h^b_i\right] \ge 1 - \varepsilon_i, \quad i \in \mathbb{Z}_{[1,n_c,]}, \tag{4}$$

where ε_i is the maximum allowed probability of constraint violation for the i^{th} constraint.

While the environment state evolution (3a) is independent of (1), the environment measurement (3b) may depend on the system states and inputs. This allows accounting for behaviors such as the dependency of the measurement on the distance or the possibility of controlling the quality of the measurement, i.e., by changing the focus of the sensors or varying the amount of sensor data processing.

B. Objective and cost function

The objective of PAC-MPC is to control (1) such that the output y^s tracks a known reference $r \in \mathbb{R}^{n_y}$, while adhering to hard constraints (2) and ICCs (4). Due to the dependence of the environment measurement (3b) on the system states and inputs, PAC-MPC may trade off the aforementioned tracking objective with actions that reduce the uncertainty in the environment, e.g., allowing for better sensing, in order to improve the future control decisions. The estimates of the mean and covariance of x^e, $\mathcal{M}^e = (\mu^e, \Sigma^e)$, evolve as

$$\mu^e_{k+1} = g_\mu\left(\mu^e_k, y^e_k, x^s_k, u^s_k\right), \tag{5a}$$
$$\Sigma^e_{k+1} = g_\Sigma\left(\Sigma^e_k, x^s_k, u^s_k\right), \tag{5b}$$

where g_μ, g_Σ updates the mean and covariance estimate of x^e based on the states and inputs of (1) and the obtained measurements of the environment y^e_k. For simplicity, the constants $\mathcal{M}^\psi$, $\mathcal{M}^\zeta$ are not explicitly shown in (5).

In prediction, we propagate the mean and covariance of x^e using a model of g, denoted as $\hat{g}$,

$$\hat{\mu}^e_{k+1} = \hat{g}_\mu\left(\hat{\mu}^e_k, \mu^y_k, x^s_k, u^s_k\right), \tag{6a}$$
$$\hat{\Sigma}^e_{k+1} = \hat{g}_\Sigma\left(\hat{\Sigma}^e_k, \Sigma^y_k, x^s_k, u^s_k\right), \tag{6b}$$

where $\hat{\mathcal{M}}^e = (\hat{\mu}^e, \hat{\Sigma}^e)$ is the *prediction* of the mean and covariance of x^e, μ^y is the predicted environment measurement, and Σ^y is the covariance of the environment measurement prediction error, $\varepsilon_y = q^e(f^e(\mu^e_{k-1}, \mu^\psi), \mu^\zeta, x^s_k, u^s_k) - y_k$, and $\mathcal{M}^y = (\mu^y, \Sigma^y)$. As in (5), we do not explicitly show $\mathcal{M}^\psi$, $\mathcal{M}^\zeta$ in (6). In what follows, we use the shorthand notations,

$$g(\mathcal{M}^e_k, y^e_k, x^s_k, u^s_k) = (g_\mu(\mu^e_k, y^e_k, x^s_k, u^s_k), g_\Sigma(\Sigma^e_k, x^s_k, u^s_k)),$$
$$\hat{g}(\mathcal{M}^{(e,y)}_k, x^s_k, u^s_k) = (\hat{g}_\mu(\mu^{(e,y)}_k, x^s_k, u^s_k), \hat{g}_\Sigma(\Sigma^{(e,y)}_k, x^s_k, u^s_k)).$$

The estimator (5) and predictor (6) play a role similar to the stabilizing gain in tube-based MPC in preventing the uncertainty, i.e., the covariance matrix, to constantly grow. In this work, we focus on the impact of the system operation on the quality of the environment measurement, that is, on the uncertainty in the environment estimate. Thus, we make the following assumption.

Assumption 1: The mean estimator (5a) and mean predictor (6a) are asymptotically convergent and unbiased for every realization of the sequence $\{(x_k, u_k)\}_k$. □

Assumption 1 ensures that the mean of the environment estimate always asymptotically converges to the true value and it is satisfied by proper estimator and predictor designs. Assumption 1 usually holds in practical implementations of environment estimation, because the system operation typically does not affect the mean of the measurement.

Due to (4), (5), the overall state includes the system state and the estimated state of the environment. Since the asymptotic convergence of μ^e is guaranteed by Assumption 1, here we consider the compound state (x^s, Σ^e).

In this paper we aim to achieve stability of both the system state x^s and the estimate covariance Σ^e. Hence, we include both in the cost function. Besides stabilizing the uncertainty, including the latter in the cost function may improve the closed-loop performance by reducing the uncertainty in future steps and, thus, reducing the tightening of the deterministic re-formulation of (4). As such, the cost function balances the achievement of the control objective with the acquisition of information about the environment through perception

$$\begin{aligned} &V_N\left(x^s_k, U_k, \Sigma^e_k, r\right) = \\ &= F(x^s_{N|k}, \Sigma^e_{N|k}, r) + \sum_{j=0}^{N-1} \ell(x^s_{j|k}, u^s_{j|k}, \Sigma^e_{j|k}, r) \\ &= F_c(x^s_{N|k}, r) + F_p(x^s_{N|k}, \Sigma^e_{N|k}, r) + \\ &\quad \sum_{j=0}^{N-1} \left[\ell_c(x^s_{j|k}, u^s_{j|k}, r) + \ell_p(x^s_{j|k}, u^s_{j|k}, \Sigma^e_{j|k}, r)\right], \end{aligned} \tag{7}$$

where r is a constant reference setpoint, $U_k = \left(u^s_{0|k}, \ldots, u^s_{N|k}\right)$ is the control sequence, $\ell_p(x, u, \Sigma, r)$, $F_p(x, \Sigma, r)$ are the *perception* (i.e., sensing) stage and terminal cost, respectively, and $\ell_c(x, u, r)$, $F_c(x, r)$ are the *control* stage and terminal cost, respectively. In (7) we do not include μ because convergence of the mean of the measurement is provided by Assumption 1.

C. Optimal Control Problem

PAC-MPC aims to solve the following optimal control problem (OCP) at each sampling time k

$$V^*_N(x^s_k, \mu^e_k, \Sigma^e_k, r) = \min_{U_k} \ V_N\left(x^s_k, U_k, \Sigma^e_k, r\right) \tag{8a}$$

$$\text{s.t. } x^s_{j+1|k} = f^s(x^s_{j|k}, u^s_{j|k}) \tag{8b}$$

$$\hat{\mathcal{M}}^e_{j+1|k} = \hat{g}(\hat{\mathcal{M}}^{(e,y)}_{j|k}, x^s_{j|k}, u^s_{j|k}) \tag{8c}$$

$$\Sigma^e_{j+1|k} = g_\Sigma(\Sigma^e_{j|k}, x^s_{j|k}, u^s_{j|k}) \tag{8d}$$

$$\hat{\mu}^y_{j|k} = \hat{q}^e(\hat{\mu}^e_{j|k}, x^s_{j|k}, u^s_{j|k}) \tag{8e}$$

$$(x^s_{j|k}, u^s_{j|k}) \in \mathcal{X} \times \mathcal{U} \tag{8f}$$

$$\mathbb{P}[h^s_i x^s_{j|k} + h^e_i x^e_{j|k} \le h^b_i] \ge 1 - \varepsilon_i, \ i \in \mathbb{Z}_{[1,n_c,]} \tag{8g}$$

$$(x^s_{N|k}, r) \in \mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k}) \tag{8h}$$

$$x^s_{0|k} = x^s_k, \ \hat{\mu}^e_{0|k} = \mu^e_k, \ \Sigma^e_{0|k} = \hat{\Sigma}^e_{0|k} = \Sigma^e_k, \tag{8i}$$

where $N \in \mathbb{Z}_+$ is the prediction horizon, and $\mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k})$ is the terminal set. The solution of the OCP is denoted by $U^*_k = (u^{s,*}_{0|k}, \ldots, u^{s,*}_{N|k})$.

In (8), there are two predictors for the environment covariance: (8d) computes Σ^e, which is used in the cost, and (8c) computes $\hat{\mathcal{M}}^e$, which is used in the ICCs (8g) and also includes uncertainty in the predicted measurement mean, predicted by $\hat{q}_e$. ICCs (4) are formulated as deterministic constraints [6], [7]

$$h^s_i x^s + h^e_i \hat{\mu}^e + [\bar{\gamma}(\hat{\Sigma}^e)]_i = h^s_i x^s + [\gamma(\hat{\mathcal{M}}^e)]_i \le h^b_i, \tag{9}$$

where $\bar{\gamma}$ is the constraint tightening due to the uncertainty on x^e and, hence, $\hat{\mathcal{M}}^e$; and γ is the effect of the environment on the constraints by its mean and covariance. At each sampling time, the MPC law takes the form

$$u^{s,*}_k = \kappa_{\text{MPC}}\left(x^s_k, \mathcal{M}^e_k, r\right) = u^{s,*}_{0|k}, \tag{10}$$

due to the receding-horizon implementation.

III. Recursive Feasibility of the PAC-MPC

Recursive feasibility of the OCP (8) is achieved by designing a terminal set $\mathcal{Z}_f(\mathcal{M}^e)$ in (8h) in which (2) and (4) are satisfied and which is positive invariant for (1) in closed-loop with a designed terminal controller $\kappa_f(x^s, \mathcal{M}^e, r)$. To achieve recursive feasibility, we make the following assumptions.

Assumption 2: Given any $\mathcal{M}^e = (\mu^e, \Sigma^e)$, for all x^s, u^s, $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, u^s)) \ge \gamma(g(\mathcal{M}^e, y^e, x^s, u^s))$. □

Assumption 3: Given $\mathcal{M}^e_1 = (\mu^e_1, \Sigma^e_1)$, $\mathcal{M}^e_2 = (\mu^e_2, \Sigma^e_2)$, such that $\gamma(\mathcal{M}^e_1) \ge \gamma(\mathcal{M}^e_2)$, for all x^s, u^s, $\gamma(\hat{g}(\mathcal{M}^{(e,y)}_1, x^s, u^s)) \ge \gamma(\hat{g}(\mathcal{M}^{(e,y)}_2, x^s, u^s))$. □

Assumption 2 ensures that the predictor does not underestimate the actual environment effect on the constraint, and can be satisfied by properly choosing the predicted measurement error Σ^y_k. Assumption 3 ensures monotonicity of the environment prediction on the constraints, and can be satisfied by the design of (5), (6).

Theorem 1: Let Assumptions 2, 3 hold, and let there exist $\kappa_f(x^s, \mathcal{M}^e, r)$, $\mathcal{Z}_f(\mathcal{M}^e)$ such that if $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$:

1) $x^s \in \mathcal{X}, \ \kappa_f(x^s, \mathcal{M}^e, r) \in \mathcal{U}$, $\mathbb{P}\left[h^s_i x^s + h^e_i x^e \le h^b_i\right] \ge 1 - \varepsilon_i, \ i \in \mathbb{Z}_{[1,n_c,]}$
2) $(f^s(x^s, \kappa_f(x^s, \mathcal{M}^e, r)), \ r) \in \mathcal{Z}_f(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, \mathcal{M}^e, r)))$.

Then, if (8) is feasible at time k for (1), (5) in closed-loop with (10) and $r_{k+1} = r_k$, (8) is feasible at time $k+1$ with probability greater or equal to $\prod_{i=1}^{n_c} \varepsilon_i$. □

Proof main steps: For Theorem 1, a feasible solution at the previous step is extended by $\kappa_f(x^s, \mathcal{M}^e, r)$. Feasibility of the ICCs is guaranteed by $\gamma(\mathcal{M}^e_{k+1}) \le \gamma(\hat{\mathcal{M}}^e_{1|k})$, by Assumption 2, and by $\gamma(\hat{\mathcal{M}}^e_{j|k+1}) \le \gamma(\hat{\mathcal{M}}^e_{j+1|k})$ for all $j \in \mathbb{Z}_{[1,N-1]}$, by combining Assumptions 2, 3. ■

While Theorem (1) ensures recursive feasibility of (8), we add a probability because the actual system constraints may be violated at the next step due to enforcing ICCs. The same holds for the subsequent results.

If the predictor (6) satisfies additional properties, a simpler design can achieve similar guarantees.

Corollary 1: Consider $\kappa_f(x^s, \mathcal{M}^e, r) = \kappa_f(x^s, r)$, $\mathcal{Z}_f(\mathcal{M}^e) = \mathcal{Z}_f(\gamma(\mathcal{M}^e))$ such that if $\gamma(\mathcal{M}^e_1) \le \gamma(\mathcal{M}^e_2)$, then $\mathcal{Z}_f(\gamma(\mathcal{M}^e_1)) \supseteq \mathcal{Z}_f(\gamma(\mathcal{M}^e_2))$. Let Assumption 2, 3 hold, and let (6) be such that $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, r))) \le \gamma(\mathcal{M}^e)$ for all $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$. Let $\kappa_f(x^s, r)$, $\mathcal{Z}_f(\mathcal{M}^e)$ be such that if $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$:

1) $x^s \in \mathcal{X}, \ \kappa_f(x^s, r) \in \mathcal{U}$, $\mathbb{P}\left[h^s_i x^s + h^e_i x^e \le h^b_i\right] \ge 1 - \varepsilon_i, \ i \in \mathbb{Z}_{[1,n_c,]}$
2) $(f^s(x^s, \kappa_f(x^s, r)), \ r) \in \mathcal{Z}_f(\mathcal{M}^e)$.

Then, if (8) is feasible at time k, for (1), (5) in closed-loop with (10) and $r_{k+1} = r_k$, (8) is feasible at time $k+1$ with probability greater or equal to $\prod_{i=1}^{n_c} \varepsilon_i$. □

Proof main steps: The proof of Corollary 1 is similar to that of Theorem 1, where we prove $\mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k}) \subseteq \mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k+1})$, which enables extending the previous solution by $\kappa_f(x^s_{N|k}, r))$. ■

The terminal controller and the terminal set can be constructed when (1) is linear,

$$x^s_{k+1} = Ax^s_k + Bu^s_k, \tag{11a}$$

$$y^s_k = Ex^s_k, \tag{11b}$$

the constraints in (2) are linear,

$$\mathcal{X} = \{x : H^x x \le K^x\}, \quad \mathcal{U} = \{u : H^u u \le K^u\}, \tag{12}$$

and the terminal controller takes the form

$$u^s = \kappa_f(x^s, r) = K_f x^s + F_f r. \tag{13}$$

Let the admissible references be constrained by $H^r r \leq K^r$, and construct the maximum output admissible set [10]

$$\mathcal{O}_\infty = \{(x_0, r_0, \eta) : \; H^x x_k \leq K^x, \; H^u \kappa_f(x,r) \leq K^u, \quad H^r r_k \leq K^r, h_i^s x_k^s + [\eta]_i \leq h_i^b, i \in \mathbb{Z}_{[1,n_c,]}, \; \forall k \in \mathbb{Z}_{0+}\}, \tag{14}$$

for (11), (13) and $r_{k+1} = r_k$, $\eta_{k+1} = \eta_k$. Define $\mathcal{O}_\infty(\eta) = \{(x^s, r) : (x^s, r, \eta) \in \mathcal{O}_\infty\}$.

Corollary 2: Consider (11) and (12), and let Assumptions 2, 3 hold. Let $\mathcal{Z}_f(\mathcal{M}^e) = \mathcal{O}_\infty(\gamma(\mathcal{M}^e))$ and (6) be such that $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, r))) \leq \gamma(\mathcal{M}^e)$ for all $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$. If (8) is feasible at time k for (11), (5) in closed-loop with (10) and $r_{k+1} = r_k$, (8) is feasible at time $k+1$ with probability greater or equal to $\prod_{i=1}^{n_c} \varepsilon_i$.□

Proof main steps: Corollary 2 is proved showing that Corollary 1 conditions holds for $\eta = \gamma(\mathcal{M}^e)$, and $\mathcal{O}^\infty(\eta_1) \supseteq \mathcal{O}^\infty(\eta_2)$, for $\eta_2 \geq \eta_1$. ■

IV. Closed-Loop Stability of the PAC-MPC

Next, we investigate conditions under which the control law (10) stabilizes (1) with the estimate provided by (5). The full state of (1), (5) is $\varphi = (x^s, \mu^e, \Sigma^e)$. However, since μ^e does not affect the dynamics and due to Assumption 1, we establish conditions for stabilizing $\xi = (x^s, \Sigma^e)$ on an equilibrium $\xi^r = (x^r, \Sigma^r)$. For ξ, we consider as norm

$$\|\xi\| = \|x^s\| + \|\Sigma^e\|_F, \tag{15}$$

which can be proved to satisfy all the properties of a norm.

For simplicity of notation, in the rest of this section we set $r = 0$ and omit it. For compactness, we denote the dynamics of (1), (5) by $\varphi_{k+1} = \Phi(\varphi_k, u_k^s, y_k^e)$. Also, we denote by ς the function that selects ξ from the full state φ, i.e., $\varsigma(\varphi) = \varsigma((x^s, \mu^e, \Sigma^e)) = (x^s, \Sigma^e) = \xi$, and $\Phi^\xi = \varsigma \circ \Phi$.

The cost function (7) can be written as

$$V_N(\xi, U) = F(\xi_{N|k}) + \sum_{j=0}^{N-1} \ell(\xi_{j|k}, u_{j|k}^s), \tag{16}$$

where $F(\xi) = F_c(x^s) + F_p(\Sigma^e)$ and $\ell(\xi, u^s) = \ell_c(x^s, u^s) + \ell_p(\Sigma^e)$, and we assume $\ell_c(x^s, 0) \leq \ell_c(x^s, u^s)$ for all u^s.

Assumption 4: There exist functions $\alpha_l^c, \alpha_l^p, \alpha_u^c, \alpha_u^p \in \mathcal{K}_\infty$ such that $\ell_c(x^s, 0) \geq \alpha_l^c(\|x^s\|)$, $F_c(x^s) \leq \alpha_u^c(\|x^s\|)$ and $\ell_p(\Sigma^e) \geq \alpha_l^p(\|\Sigma^e\|_F)$, $F_p(\Sigma^e) \leq \alpha_u^p(\|\Sigma^e\|_F)$. □

Assumption 5: There exists $u = \kappa_f(\varphi)$, such that for all $x^s \in \mathcal{Z}_f(\mathcal{M}^e)$, $F(\xi) \geq \ell(\xi, \kappa_f(\varphi)) + F(\Phi^\xi(\varphi, \kappa_f(\varphi), y^e))$ and $f^s(x^s, \kappa_f(\varphi)) \in \mathcal{Z}_f(\hat{g}(\mathcal{M}_k^{(e,y)}, x^s, \kappa_f(\varphi)))$. □

Under Assumptions 4, 5, the value function of (8) is a Lyapunov function for the closed-loop system.

Lemma 1: Let Assumptions 4, 5 hold, then there exist functions $\alpha_l, \alpha_u, \alpha_\Delta \in \mathcal{K}_\infty$ such that

$$\alpha_l(\|\xi\|) \leq V_N^*(\varphi) \leq \alpha_u(\|\xi\|) \tag{17a}$$

$$V_N^*(\Phi(\varphi, \kappa_{\text{MPC}}(\varphi), y^e)) - V_N^*(\varphi) \leq -\alpha_\Delta(\|\xi\|), \tag{17b}$$

when (8) is feasible for $(x_k^s, \mu_k^e, \Sigma_k^e) = \varphi$ and for $(x_k^s, \mu_k^e, \Sigma_k^e) = \Phi(\varphi, \kappa_{\text{MPC}}(\varphi), y^e)$. □

Proof main steps: Lemma 1 is proved by deriving α_l, α_u from α_l^c, α_l^p, α_u^c, α_u^p based on Assumption 4, the properties of class-$\mathcal{K}$ functions and (15), and α_Δ from Assumption 5. ■

The terminal cost design is then obtained as follows.

Theorem 2: Let Assumption 4 and the conditions of Theorem 1 or Corollary 1 hold. If for all $x_s \in \mathcal{Z}_f(\mathcal{M}^e)$

$$F_c(f(x^s, \kappa_f(\varphi)) - F_c(x^s) + \ell_c(x^s, \kappa_f(\varphi)) \leq -M(x^s) \tag{18a}$$

$$F_p(\hat{g}_\Sigma(\Sigma^{(e,y)}, \kappa_f(\xi), x^s, \kappa_f(\varphi))) - F_p(\Sigma_e) + \ell_p(\Sigma_e) \leq M(x^s), \tag{18b}$$

where M is a nonnegative function, then at every step the closed-loop (1), (5), (10) has probability at least $\prod_{i=1}^{n_c} \varepsilon_i$ to evolve satisfying the Lyapunov function (17). □

Proof main steps: For Theorem 2, recursive feasibility with probability $\prod_{i=1}^{n_c} \varepsilon_i$, i.e., the satisfaction of the constraints enforced by ICC at the next step, is provided by Theorem 1 or Corollary 1, while (18) satisfies the assumptions in Lemma 1. ■

The stability of φ requires also the stability of the mean estimate, which is provided by the estimator (5a), according to Assumption 1.

A. Constructive Design for Linear Case

Consider (11), (12), and linear environment dynamics

$$x_{k+1}^e = A^e x_k^e + B^e \psi_k \tag{19a}$$

$$y_k^e = C^e(x_k^s, u_k^s) x_k^e + D^e(x_k^s, u_k^s) \zeta_k, \tag{19b}$$

where $\psi_k \sim \mathcal{N}(\mu^\psi, \Sigma^\psi)$ and $\zeta_k \sim \mathcal{N}(\mu^\zeta, \Sigma^\zeta)$. The measurement equation (19b) depends on system states and inputs to account for a variable perception quality.

The estimator (5) is naturally selected as a Luenberger-type estimator with update equations

$$\mu_{k+1}^e = \Lambda_k \mu_k^e + B^e \mu^\psi - L_k y_k^e, \tag{20a}$$

$$\Sigma_{k+1}^e = \Lambda_k \Sigma_k^e \Lambda_k^\top + Q + R_k, \tag{20b}$$

where $\Lambda_k = \Lambda(x_k^s, u_k^s) = A^e + L_k C_k$, $L_k = L(x_k^s, u_k^s)$, $C_k = C^e(x_k^s, u_k^s)$, $Q = B^e \Sigma^\psi B^{e\top}$, and $R(x_k^s, u_k^s) = L_k D_k^e \Sigma^\zeta (L_k D_k^e)^\top$, $D_k^e = D^e(x_k^s, u_k^s)$. The same equations are used for predictor (6), where μ_k^y replaces y_k^e in (20a), and $\hat{R}(x_k^s, u_k^s) = L_k(D_k^e \Sigma^\zeta D_k^{e\top} + \Sigma_k^y) L_k^\top$ in (20b), i.e., includes a term for the measurement prediction error.

In (16), we choose as control stage and terminal costs

$$\ell_c(x, u, r) = ||x - r^x||_{Q_c}^2 + ||u - r^u||_{R_c}^2, \tag{21a}$$

$$F_c(x, r) = ||x - r^x||_{P_c}^2, \tag{21b}$$

with $Q_c, R_c, P_c > 0$, and r^x and r^u the reference setpoints of x^s and u^s, respectively. The equilibrium for the uncertainty covariance is Σ^r, the solution of the Lyapunov equation $\Sigma^r = \Lambda(r^x, r^u) \Sigma^r \Lambda(r^x, r^u)^\top + Q + R(r^x, r^u)$, which is computed with steady-state estimator gain $L(r^x, r^u)$. In (16), we choose as perception stage cost and terminal cost

$$\ell_p(\Sigma_k^e) = S_c \left|\left|\bar{\Sigma}_k^e\right|\right|_F^2, \quad F_p(\Sigma_k^e) = W_c \left|\left|\bar{\Sigma}_k^e\right|\right|_F^2, \tag{22}$$

where $S_c, W_c \in \mathbb{R}_+$ are weights and $\bar{\Sigma}^e = \Sigma^e - \Sigma^r$.

Corollary 3: Consider (11), (12), (13), (19), (20), and $\mathcal{Z}_f(\mathcal{M}^e) = \mathcal{O}_\infty(\gamma(\mathcal{M}^e))$. Given $r_k = r$ for all $k \in \mathbb{Z}_{0+}$, let the assumptions of Corollary 2 hold, $\bar{R}_k = R(x^s_k, u^s_k) - R(r^x, r^u)$, with steady state estimator gain $L(r^x, r^u)$, and $\vartheta_\Lambda = \max_{x^s:(x^s,r)\in\mathcal{Z}_f(\mathcal{M}^e)} ||\Lambda(x^s, \kappa_f(x_s, r))||^2_F$. Given $Q_c, R_c > 0$, if there exist $P_c > 0$, W_c, S_c, $\rho \in \mathbb{R}_+$, $M_c \geq 0$, such that for all $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$,

$$x^{s\top} M_c x^s \geq W_c(1+\rho)\left|\left|\bar{R}(x^s, \kappa_f(x_s, r))\right|\right|^2_F, \tag{23a}$$

$$S_c \leq W_c\left[1 - \left(1+\rho^{-1}\right)\vartheta^2_\Lambda\right], \tag{23b}$$

$$P_c \geq K_f^\top R_c K_f + (Q_c + M_c) + (A^s + B^s K_f)^\top P_c (A^s + B^s K_f), \tag{23c}$$

the evolution of $\xi = (x^s, \Sigma^e)$ for (11), (12), (13), (19), (20) satisfies (17) at each step with probability $\prod_{i=1}^{n_c} \varepsilon_i$. □

Proof main steps: The terminal controller κ_f satisfies Corollary 2, and hence achieves the properties in Corollary 1. Inequality (18b) is shown to hold by (23a), (23b), by using Young's inequality, the properties of the Frobenious norm, and linear algebra. Condition (18a) holds for M_c when (23c) is satisfied. Hence, all the assumptions of Theorem 2 hold and as a consequence the result of Theorem 2 hold, so that the statement of this corollary also holds. ■

Condition (23a) bounds the perception cost increase due to the difference between R_k and the steady state, and condition (23c) ensures that such an increase is smaller than the control cost decrease. Condition, (23b) simply determines the relative weight of perception terminal and stage cost.

V. Numerical Example

We demonstrate the performance of PAC-MPC on a simple double integrator system, which enables visualizing the results. A more realistic case study in automated driving, including rationales for the perception quality models, was shown in [5]. The system model takes the form of (11), where $A = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix}1/2 & 0\\ 1 & 0\end{smallmatrix}\right]$, $E = [\,1\ 0\,]$, and $[u^s]_2 \in [0, 1]$ is a perception input that only affects the measurement quality. The environment model is (19), where $A^e = I_2$, $B^e = 0$, $C^e = I_2$, $D^e_k = (1 - \beta[u^s_k]_2)\bar{D}$, with $\beta \in (0, 1)$, $\bar{D} \in \mathbb{R}^{2\times 2}$. Therefore, $[u^s]_2$ directly affects the measurement quality, as evidenced by D^e_k. The random vector ζ follows a standard normal distribution. The constraint sets in (2) are $\mathcal{X} = \{x \in \mathbb{R}^2 : \ |[x]_i| \leq 10,\ i = 1, 2\}$, $\mathcal{U} = \{u \in \mathbb{R}^2 : , [u]_1 \in [-5, 5],\ [u]_2 \in [0, 1]\}$. System (11) is subject to $[x^s]_i - [x^e]_i \leq 0$, $i = 1, 2$, formulated as ICCs (4), for $\varepsilon_i = 0.05$. We solve (8) formulated in CasADi using IPOPT. Process and measurement noise are randomly sampled according to their distributions.

Fig. 1 shows the closed-loop state and input trajectories for different initial system states and fixed initial environment uncertainty. Initially, the environment uncertainty makes the setpoint of $[x^s]_1$ unreachable. Therefore, PAC-MPC uses the perception command $[u^s]_2$ to improve the measurement quality, so that it can reduce the environment uncertainty more quickly. The trajectories satisfy the constraints according to (4) and stabilize to their steady-state, as expected. Fig. 1 shows that the closed-loop trajectory of the environment covariance has low sensitivity to the initial system state in this case of input-dependent measurement quality.

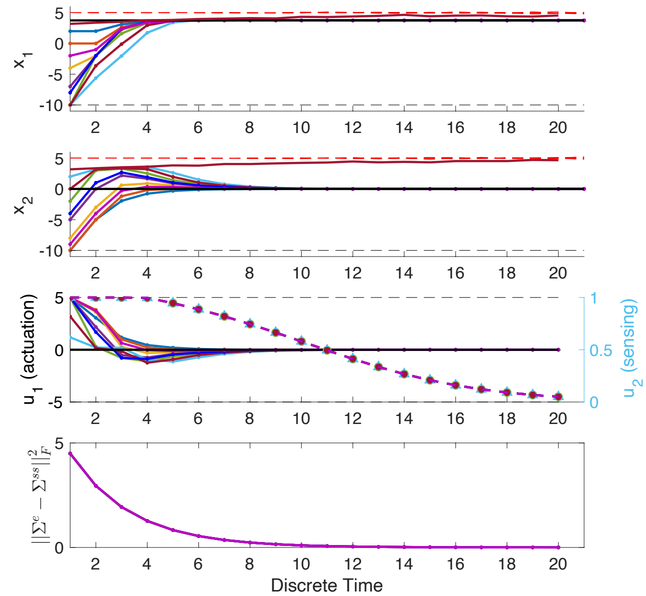

Fig. 1. PAC-MPC simulations with input-dependent measurement quality. Closed-loop state, input, and environment covariance trajectories for different initial system states and fixed initial environment uncertainty. Environment state x^e (dash, red), constraints due to environment mean and covariance, i.e., $-\gamma(\mathcal{M}^e)$, (solid, dark red), setpoint r^x, r^u (solid, black), deterministic constraints (dash, black), perception input (dash, cyan axis).

Fig. 2 shows the phase plot of the closed-loop state trajectories, including with the region where the initial state is recursively feasible for the initial environment uncertainty. As time progresses the trajectories leave this region because the uncertainty has been reduced and, hence, a larger area of the state space has become recursively feasible.

Fig. 3 shows the same simulation as in Fig. 1, for fixed initial system states and different initial environment uncertainty. The plot shows some brief constraint violations

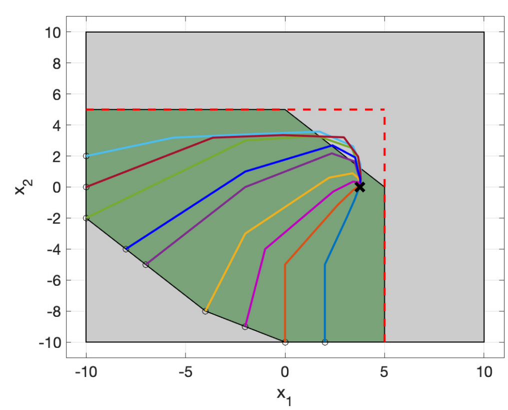

Fig. 2. PAC-MPC simulations with input-dependent measurement quality. Phase plot of closed-loop state trajectories for different initial system states and fixed initial environment uncertainty. Constraints due to the actual environment state x^e (dash, red), initial region of recursive feasibility (green), setpoint (cross, black).

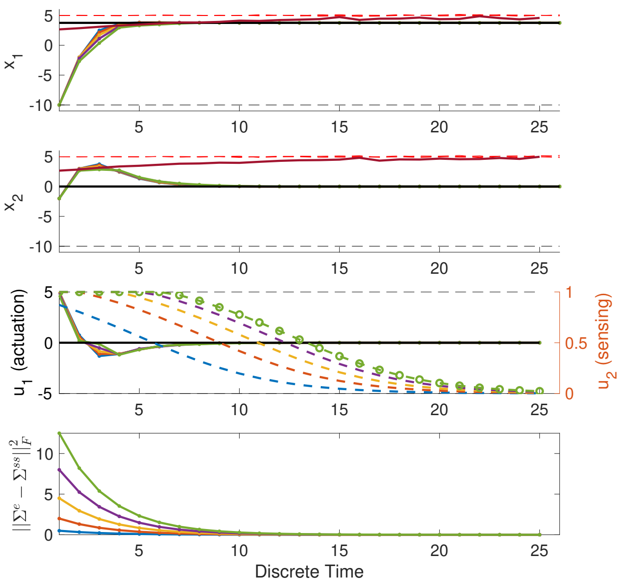

Fig. 3. PAC-MPC simulations with input-dependent measurement quality. Closed-loop state, input, and covariance trajectories for different initial environment uncertainties and fixed initial system state. Environment state x^e (dash, red), constraints due to environment mean and covariance, i.e., $-\gamma(\mathcal{M}^e)$, (solid, dark red), setpoint r^x, r^u (solid, black), deterministic constraints (dash, black), perception input (dash, orange axis).

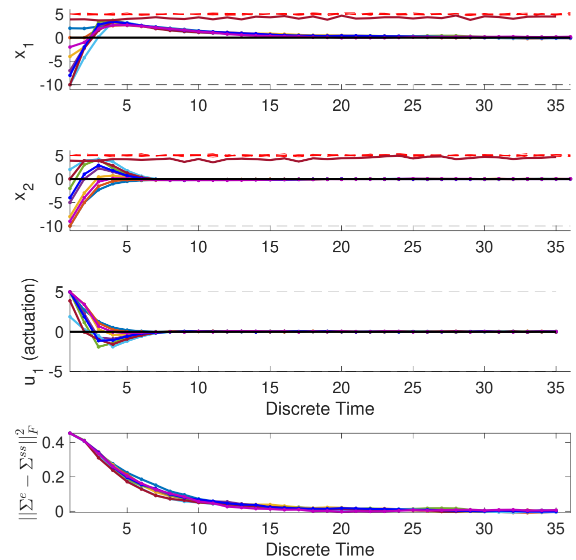

Fig. 4. PAC-MPC simulations with state-dependent measurement quality. Closed-loop state and input trajectories for different initial system states and fixed initial environment uncertainty. Environment state x^e (dash, red), constraints due to environment mean and covariance, i.e., $-\gamma(\mathcal{M}^e)$, (solid, dark red), setpoint r^x, r^u (solid, black), deterministic constraints (dash, black).

according to (4), and asymptotic stability to the setpoint.

Next, for the same system dynamics, we show a case of state-dependent measurement inspired from [11], where $D^e_k = \bar{D}\,||\left([x^s_k]_1 - [x^e_k]_1\right)/\ell_L||^\beta$, ℓ_L is a length-scale constant, and $\beta > 0$ determines the rate of deterioration as function of the distance. In this example, the perception command $[u^s]_2$ is also removed, i.e., $u^s = [u^s]_1 \in \mathbb{R}$. Fig. 4 shows the closed-loop trajectories, where the state trajectories overshoot the setpoint to reduce the distance from x^e, which then reduces the environment covariance, and eventually stabilize both the system state and the environment covariance to their setpoints. The environment covariance sensitivity to the initial conditions of the system state is increased with respect to that in Fig. 1 since now the measurement uncertainty depends directly on the system trajectory, as opposed to a dedicated sensing input.

VI. Conclusions

We studied the stability properties of PAC-MPC that controls a known system in a partially unknown environment being estimated, where the estimate is affected by the system operation. We derived conditions for general nonlinear dynamics and a constructive procedure when these are specialized to the linear case. In the future, we will investigate reducing conservativeness in accounting for the measurement prediction error.

References

[1] S. Di Cairano and I. V. Kolmanovsky, "Real-time optimization and model predictive control for aerospace and automotive applications," in *Proc. American Control Conf.*, 2018, pp. 2392–2409.

[2] F. Bourgault, A. A. Makarenko, S. B. Williams, B. Grocholsky, and H. F. Durrant-Whyte, "Information based adaptive robotic exploration," *IEEE Int. Conf. Intelligent Robots and Systems*, 2002.

[3] N. E. Du Toit and J. W. Burdick, "Robot motion planning in dynamic, uncertain environments," *IEEE Trans. on Robotics*, vol. 28, no. 1, pp. 101–115, 2011.

[4] D. Falanga, P. Foehn, P. Lu, and D. Scaramuzza, "PAMPC: Perception-aware model predictive control for quadrotors," in *IEEE Int. Conf. Intelligent Robots and Systems*, 2018.

[5] A. D. Bonzanini, A. Mesbah, and S. Di Cairano, "Perception-aware chance-constrained model predictive control for uncertain environments," in *Proceedings of the American Control Conference*, New Orleans, LA, 2021.

[6] M. Farina, L. Giulioni, and R. Scattolini, "Stochastic linear model predictive control with chance constraints–a review," *Jour. Process Control*, vol. 44, pp. 53–67, 2016.

[7] A. Mesbah, I. V. Kolmanovsky, and S. Di Cairano, "Stochastic Model Predictive Control," in *Handbook of Model Predictive Control*, 2019, pp. 75–97.

[8] T. A. N. Heirung, B. E. Ydstie, and B. Foss, "Dual adaptive model predictive control," *Automatica*, vol. 80, pp. 340–348, 2017.

[9] A. Mesbah, "Stochastic model predictive control with active uncertainty learning: A survey on dual control," *Annual Reviews in Control*, vol. 45, pp. 107–117, 2018.

[10] E. Garone, S. Di Cairano, and I. Kolmanovsky, "Reference and command governors for systems with constraints: A survey on theory and applications," *Automatica*, vol. 75, pp. 306–328, 2017.

[11] B. Huang, L. Xie, and Z. Yang, "TDOA-based source localization with distance-dependent noises," *IEEE Transactions on Wireless Communications*, vol. 14, no. 1, pp. 468–480, 2014.